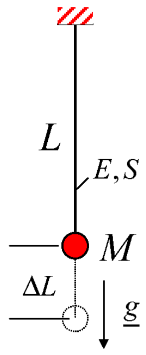
Exercise 1 – Suspended mass

- Single-element discrete model: check vs. analytical solution
- Explain possible reasons for observed discrepancies
- Try out different values: e.g. gravity 1000 times smaller
- Discuss multi-element discrete model



$L = 1$ m
$M = 100$ kg
$g = 10$ m/s^2
$S = 2.5\times10^{-8}$ m^2
$E = 2\times10^{11}$ Pa
$\rho = 8000$ kg/m^3
$\nu = 0$

- Replace gravity by initial velocity and discuss effects of structure modeling: A) as a bar, B) as a cable …

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Linear static analysis

The applied force is:

$$F = mg = 1000 \text{ N}$$

and the stress:

$$\sigma = F/S = 1000/2.5\times10^{-8} = 4\times10^{10} \text{ Pa}$$

This induces a strain:

$$\varepsilon = \sigma/E = 4\times10^{10}/2\times10^{11} = 0.2$$

The elongation is:

$$\Delta L = L\varepsilon = 0.2 \text{ m}$$

Linear dynamic analysis

The cable mass is:

$$m = \rho V = \rho L S = 8000 \cdot 1 \cdot 2.5\times10^{-8} \ll M$$

and is therefore negligible with respect to the concentrated mass M .

By schematizing this system as a single-d.o.f. oscillator, the pulsation of the oscillation would be:

$$\omega = 2\pi f = \sqrt{K/M}$$

where f indicates the frequency and K the stiffness. The latter is given by:

$$K = \frac{F}{\Delta L} = \frac{\sigma S}{\Delta L} = \frac{E\varepsilon S}{\Delta L} = \frac{E\Delta L}{L\Delta L} S = \frac{ES}{L}$$

Hence:

$$\omega = \sqrt{\frac{ES}{LM}} = \sqrt{\frac{2\times 10^{11}\cdot 2.5\times 10^{-8}}{1\cdot 100}} = 7.071\ \mathrm{s}^{-1}$$

or, in terms of frequency:

$$f = \frac{\omega}{2\pi} = 1.125\ \mathrm{Hz}$$

The period is:

$$T = 1/f = 0.889\ \mathrm{s}$$

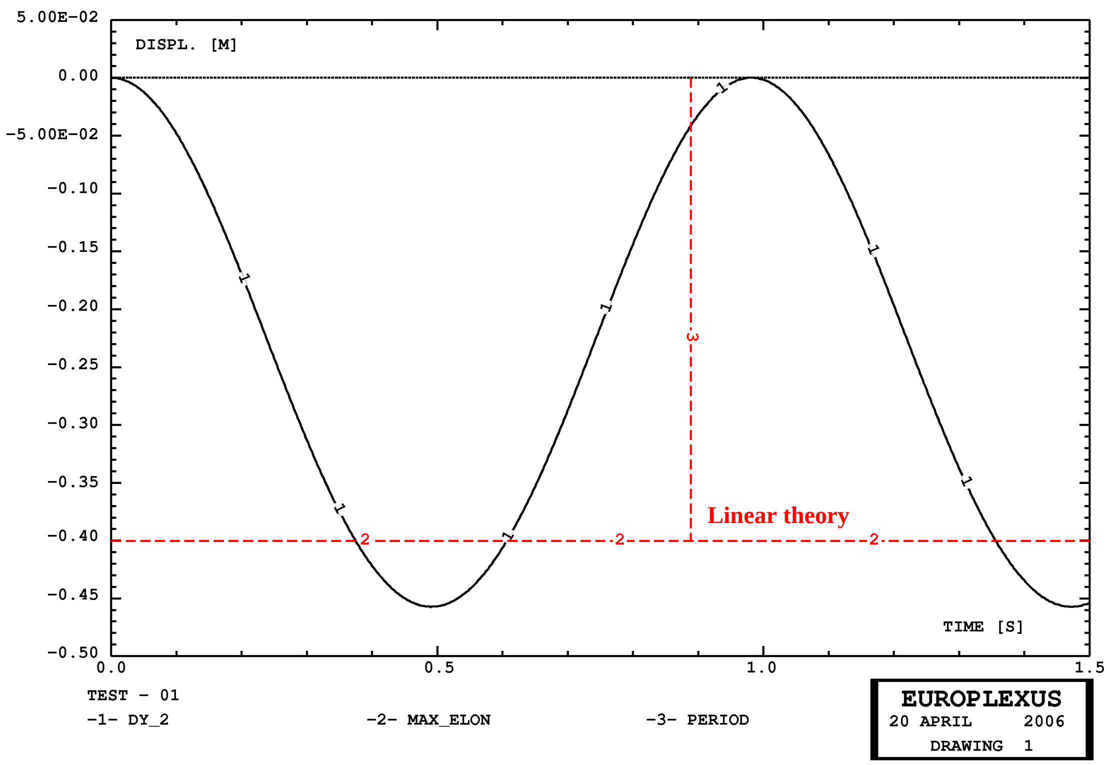
The expected behaviour is a sinusoidal oscillation around the static deflection value. The maximum dynamic deflection is twice the static value:

$$\Delta L^{\mathrm{dyn}} = 2\cdot \Delta L^{\mathrm{sta}} = 0.4\ \mathrm{m}$$

<u>Numerical simulation</u>

TEST01

Discretize the system by just one Finite Element of the "cable" type (FUN2). These elements do not offer any resistance to bending. In addition, the assumed material (of type FUNE) is linear elastic but with no resistance to compression (only to traction) and the formulation is large-strain as concerns axial deformations. The result is:



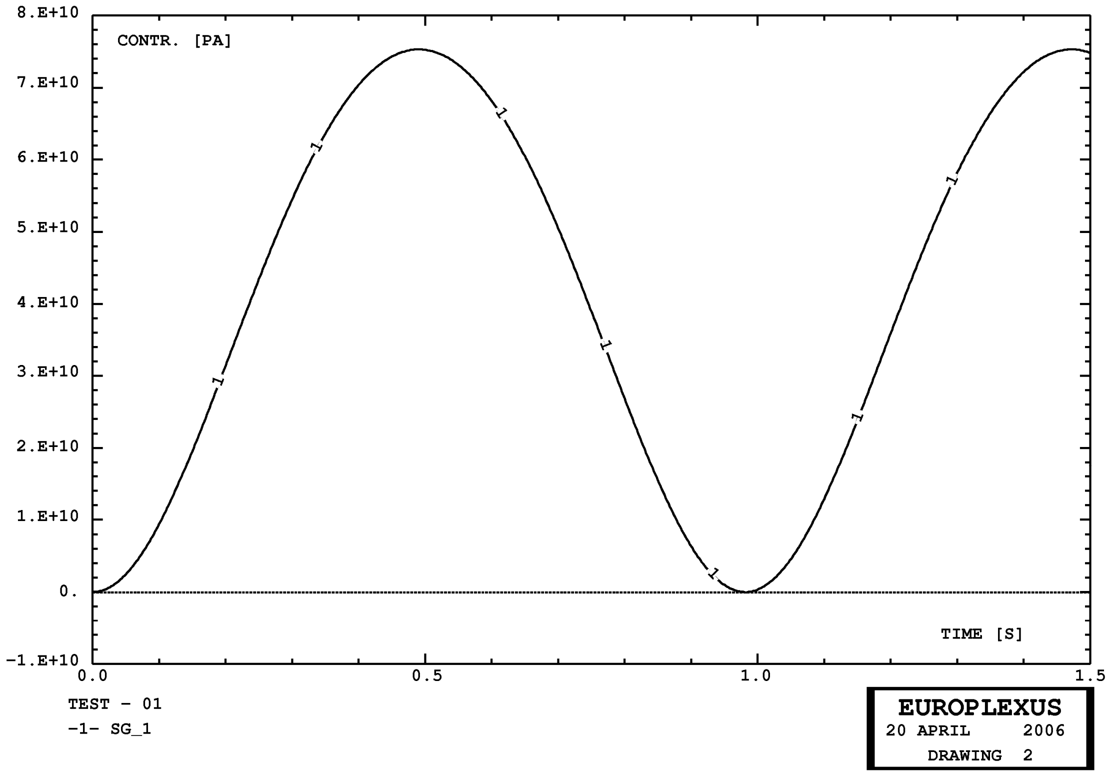
The obtained displacement resembles the expected one. However:

- The obtained maximum elongation is larger than the expected value (~0.45 instead of 0.40)
- The oscillation period is longer than expected (~0.98 instead of 0.89).

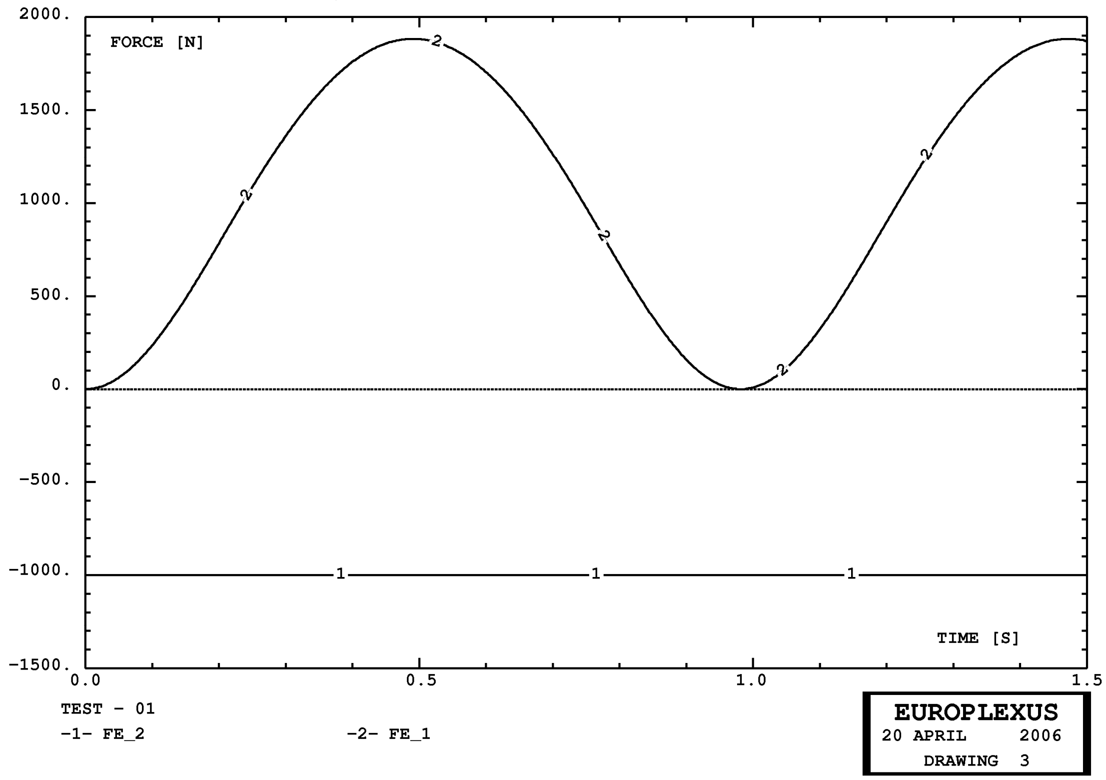
The EUROPLEXUS input file for this problem is:

```
TEST - 01
*-----------------------------------------------------------------------
ECHO
*CONV win
*------------------------------------------------------------Problem type
CPLA NONL LAGR
*------------------------------------------------------------Dimensioning
DIME
  PT2L 2 FUN2 1 PMAT 1 ZONE 2
  TABL 1 2
  FORC 1
TERM
*----------------------------------------------------------------Geometry
GEOM LIBR POIN 2 FUN2 1 PMAT 1 TERM
 0 0  0 -1
 1 2
 2
*---------------------------------------------------Geometrical complements
COMP EPAI 2.5E-8 LECT 1 TERM
*-----------------------------------------------------------Material data
MATE FUNE RO 8000. YOUN 2.0E11 NU 0.0 ELAS 2.0E11 ERUP 1.0E0
          TRAC 1 2.0E11 1.E0
          LECT 1 TERM
     MASS 100.0 LECT 2 TERM
*-----------------------------------------------------Boundary conditions
LINK COUP
     BLOQ 2 LECT 1 TERM
*-----------------------------------------------------------Applied loads
CHAR 1 FACT 2 FORC 2 -1.E3 LECT 2 TERM
       TABL 2 0.0 1.0 10.0 1.0
*-----------------------------------------------------------------Outputs
ECRI DEPL VITE CONT ECRO TFREQ 0.5
     FICH ALIC TEMP FREQ 20
               POIN LECT 1 2 TERM
               ELEM LECT 1 TERM
*-----------------------------------------------------------------Options
OPTI PAS UTIL NOTEST
*-----------------------------------------------------Transient calculation
CALCUL TINI 0. TEND 1.5 PASF 0.1E-3
*=========================================================POST-TREATMENT
SUIT
Post-treatment
ECHO
RESU ALIC TEMP GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
*-----------------------------------------------------Curve definitions
COUR 1 'dy_2'     DEPL COMP 2 NOEU LECT  2 TERM
COUR 2 'sg_1'     CONT COMP 1 ELEM LECT  1 TERM
COUR 3 'fe_2'     FORC COMP 2 NOEU LECT  2 TERM
COUR 4 'fe_1'     FORC COMP 2 NOEU LECT  1 TERM
DCOU 6 'Max_elon' 2
       0.0 -0.4
       1.5 -0.4
DCOU 7 'Period' 2
       0.889 0.0
       0.889 -0.4
*-------------------------------------------------------------------Plots
trac 1 6 7 axes 1.0 'DISPL. [M]' yzer
           colo noir roug roug
           dash 0    2    2
trac 2 axes 1.0 'CONTR. [PA]' yzer
trac 3 4 axes 1.0 'FORCE [N]' yzer
list 1 axes 1.0 'DISPL. [M]'
*-----------------------------------------------Results qualification
QUAL DEPL COMP 2 LECT   2 TERM REFE -4.53636E-1  TOLE 1.E-2
     CONT COMP 1 LECT   1 TERM REFE  7.48171E+10 TOLE 1.E-2
*=======================================================================
FIN
```

The stress history is:



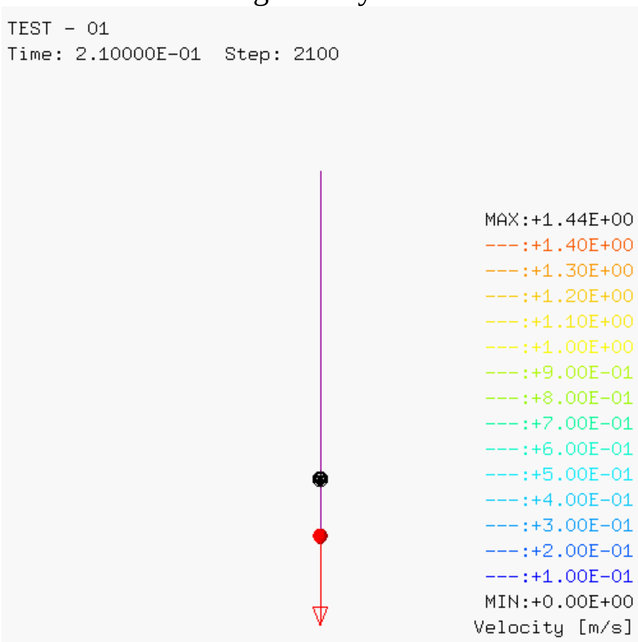
Note that the maximum stress is slightly lower than twice the static value ($\sim 7.5\times10^{10}$ Pa instead of 8.0×10^{10} Pa). The external forces at the two extremities are:



Again, the maximum reaction force at the fixed extremity is slightly lower than twice the applied force (~1900 N instead of 2000 N).

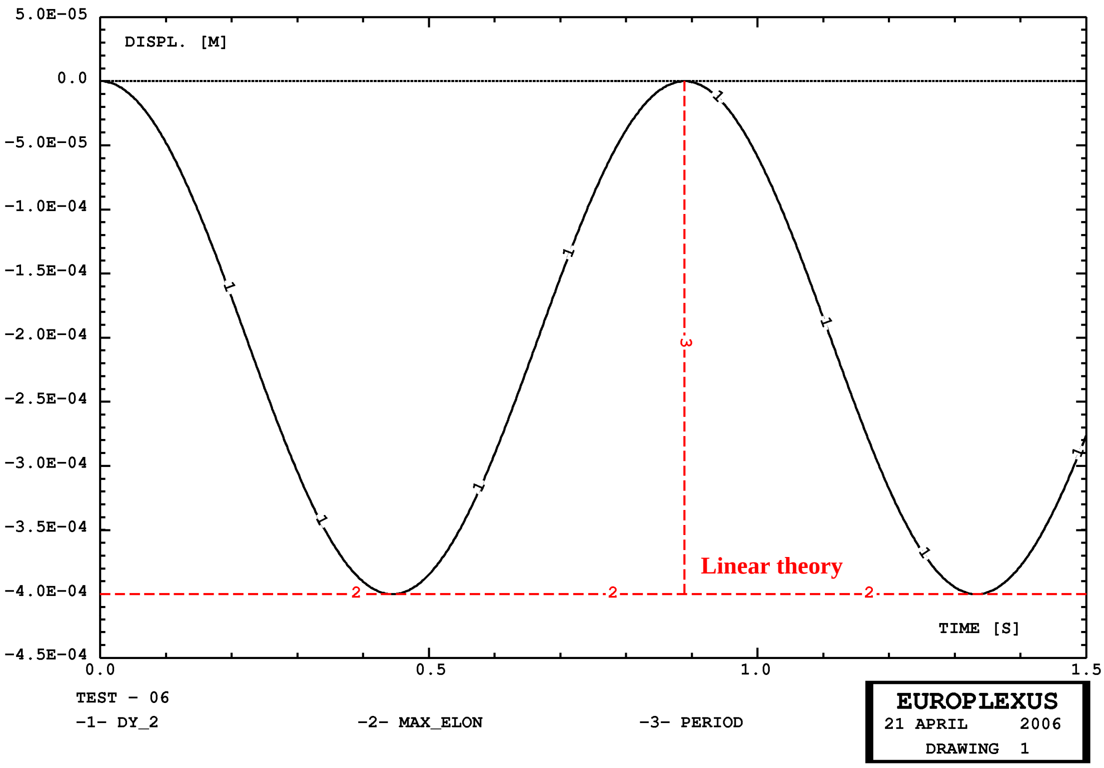
These discrepancies are due to the fact that the code uses a large-strain formulation for the cable element, and that for the chosen values of the parameters the obtained deformations are indeed quite large.

Here is an animation of the deformed geometry:

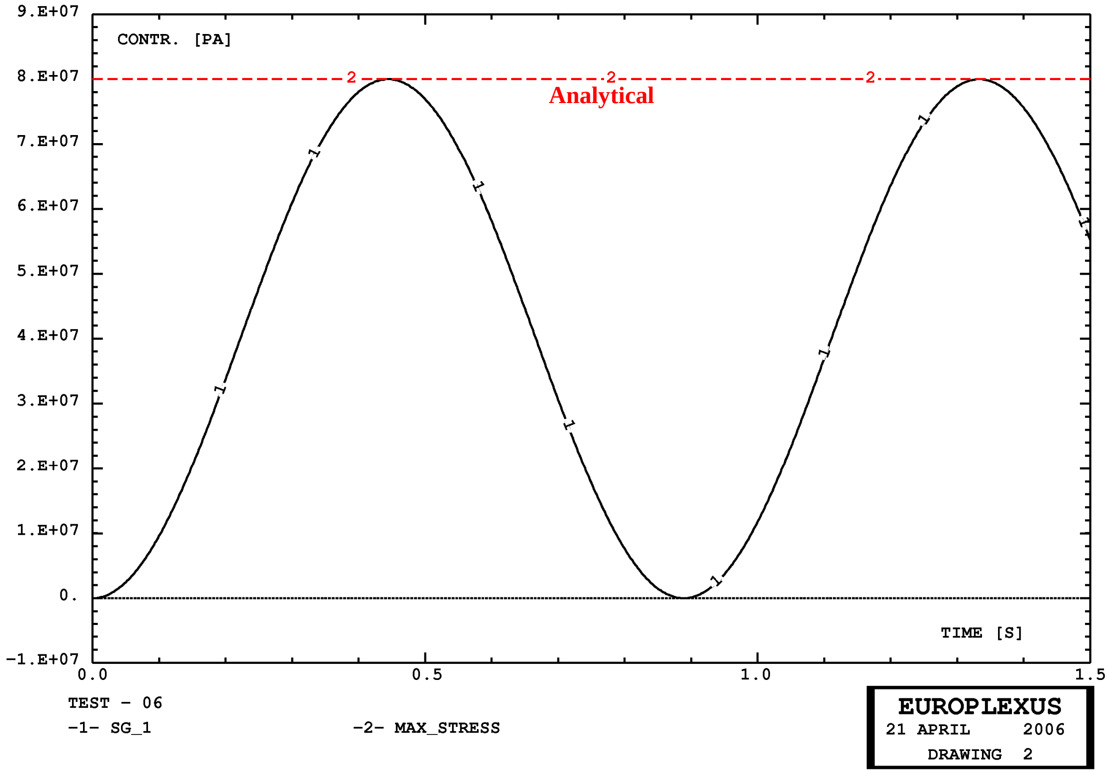


TEST06

Same as TEST01 but uses an applied load 1000 times smaller, so as to remain in the small-strain range. The results are in good agreement with the linear analysis. The maximum deflection is 4×10^{-4}:



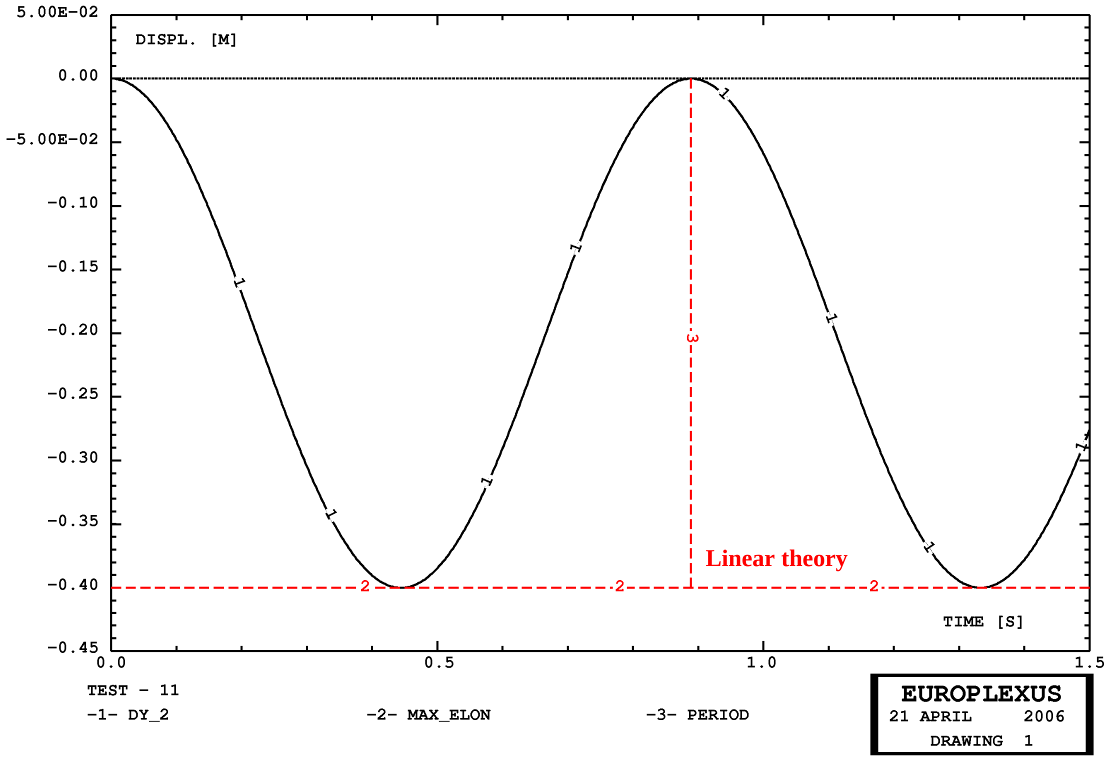
and the oscillation period is very close to the analytical value of 0.889 s. The maximum stress is 8×10^{7}:



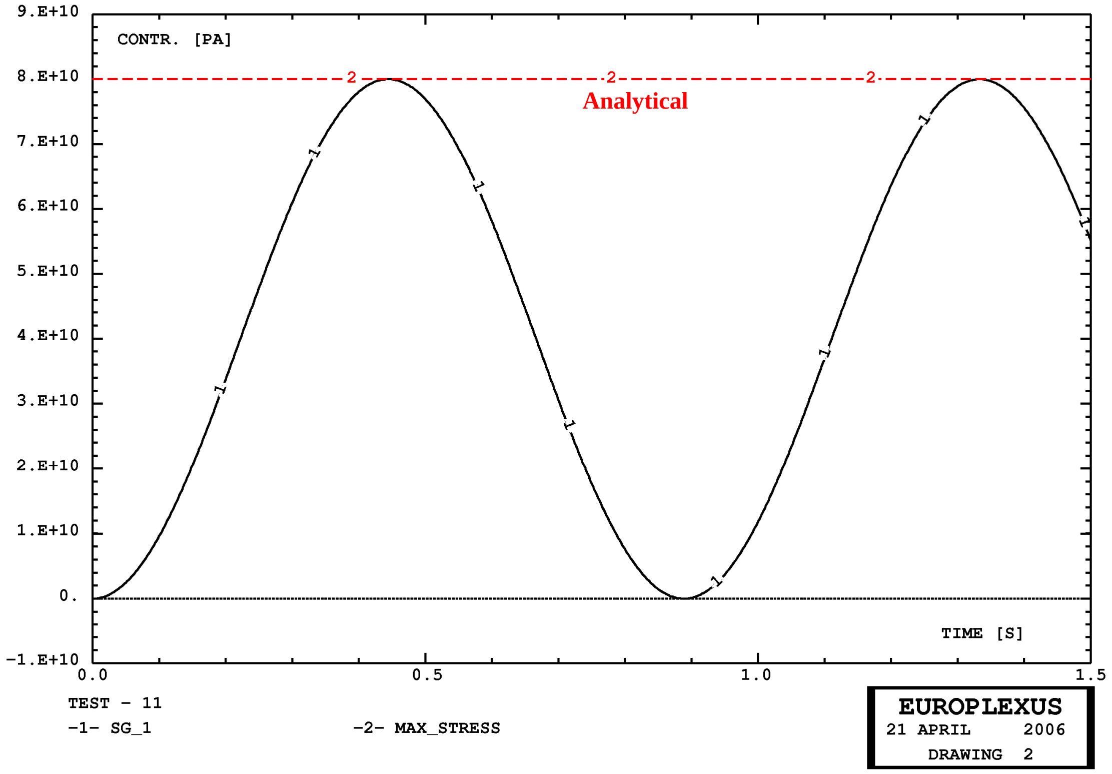
TEST11

Same as TEST01 but use a special option (`OPTI EDSS`) that causes the cable element to use a small-strain formulation instead of the default large-strain formulation. The only difference is in the calculation of the axial strain increment, which reads $\Delta\varepsilon = \Delta L / L_0$ instead of $\Delta\varepsilon = \Delta L / L$.

The results are in good agreement with the linear analysis. The maximum deflection is 0.4 and the period is 0.889 s:



The maximum stress is 8×10^{10}:



TEST08

Same as TEST01 but uses a special option to obtain a “quasi-static” calculation (`OPTI QUAS STAT FSYS beta`). The code adds a linear damping represented by an external force proportional to the mass for each degree of freedom:

$$F_{qs} = -4\pi\beta f_{sys} m v$$

where m is the mass and v the velocity. In practice, only the product βf_{sys} is relevant. The case $\beta = 1$ corresponds to critical damping for the frequency f_{sys}. Here we use $\beta = 1$ and $f_{sys} = 1$ Hz, because in the large-strain case we saw in calculation TEST01 that the obtained period is 0.98 so the frequency is $1/0.98 \approx 1$.

The resulting displacement at 1.5 s is 0.221345, which is already very close to the expected value of 0.221403 (i.e. $\exp(0.2)-1$) that would result from a large-strain static analysis (see below).

Explanation: since $\nu = 0$ there is no cross-section variation as long as the material remains linear elastic (as assumed here). The Cauchy stress is therefore:
$\sigma = F / S_0 = 1000 \text{ N} / 2.5 \times 10^{8} \text{ m}^2 = 4 \times 10^{10} \text{ Pa}$.
This induces a strain (natural): $e = \sigma / E = 4 \times 10^{10} \text{ Pa} / 2 \times 10^{11} \text{ Pa} = 0.2$. But we have $e = \ln\ (L / L_0)$ so that $L / L_0 = \exp\ (e)$ and $L = L_0 \exp\ (e)$.
The elongation (or the displacement) is therefore: $\Delta L = L - L_0 = L_0 \exp\ (e) - L_0$. Since $L_0 = 1$ in this example, we get finally:

$$\Delta L = \exp(0.2) - 1 = 0.221403$$

Comparison of natural vs. engineering strain

1) Engineering stress is defined as:

$$\sigma \doteq F / S_0$$

and engineering strain as:

$$\varepsilon \doteq \int_{L_0}^{L} \frac{1}{L_0} dl = \frac{\Delta L}{L_0} = \frac{L - L_0}{L_0} = \frac{L}{L_0} - 1$$

2) Natural (Cauchy) stress is defined as:

$$s \doteq F / S$$

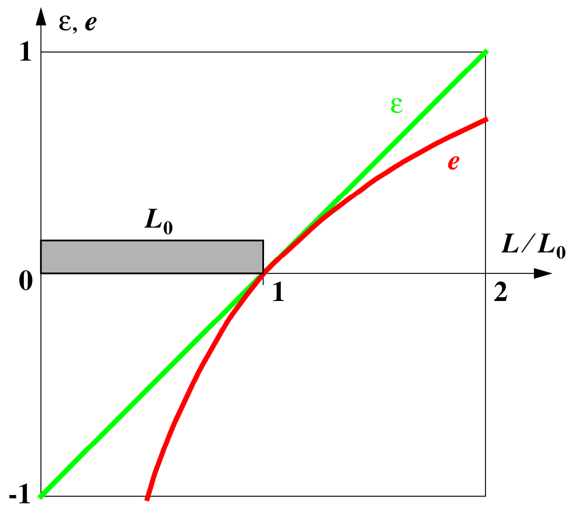
and natural strain as:

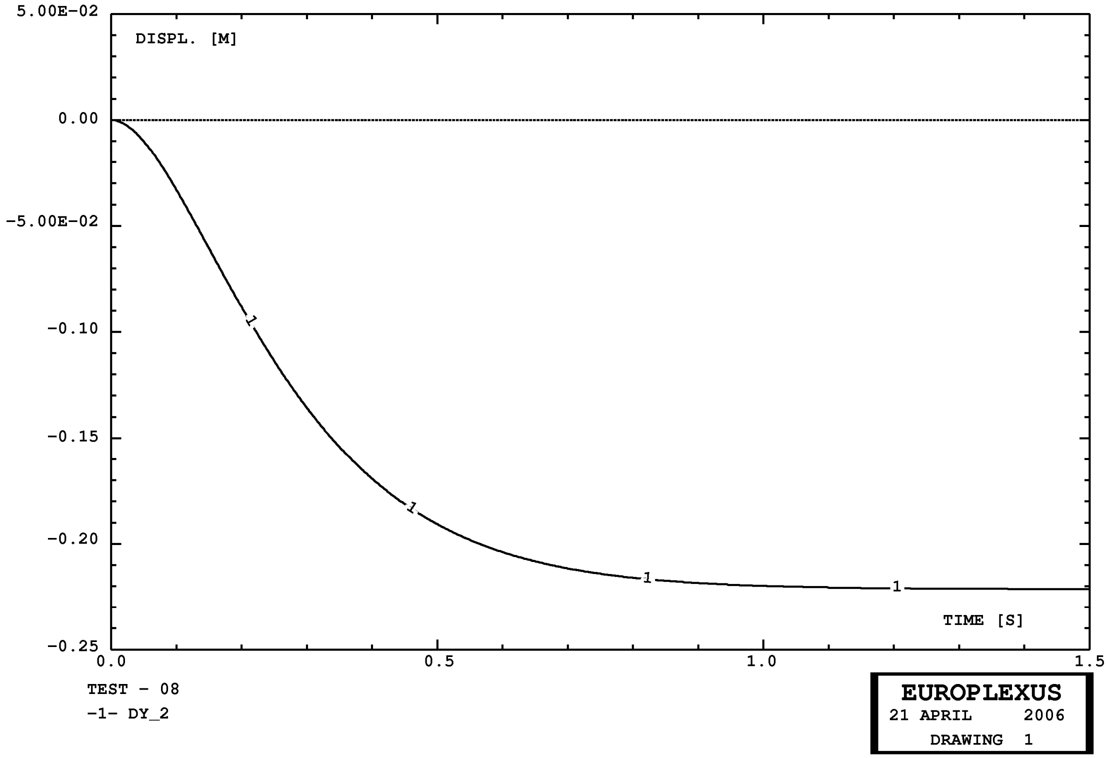
$$e \doteq \int_{L_0}^{L} \frac{1}{L} dl = \ln \frac{L}{L_0}$$

3) We have thus the relationships:

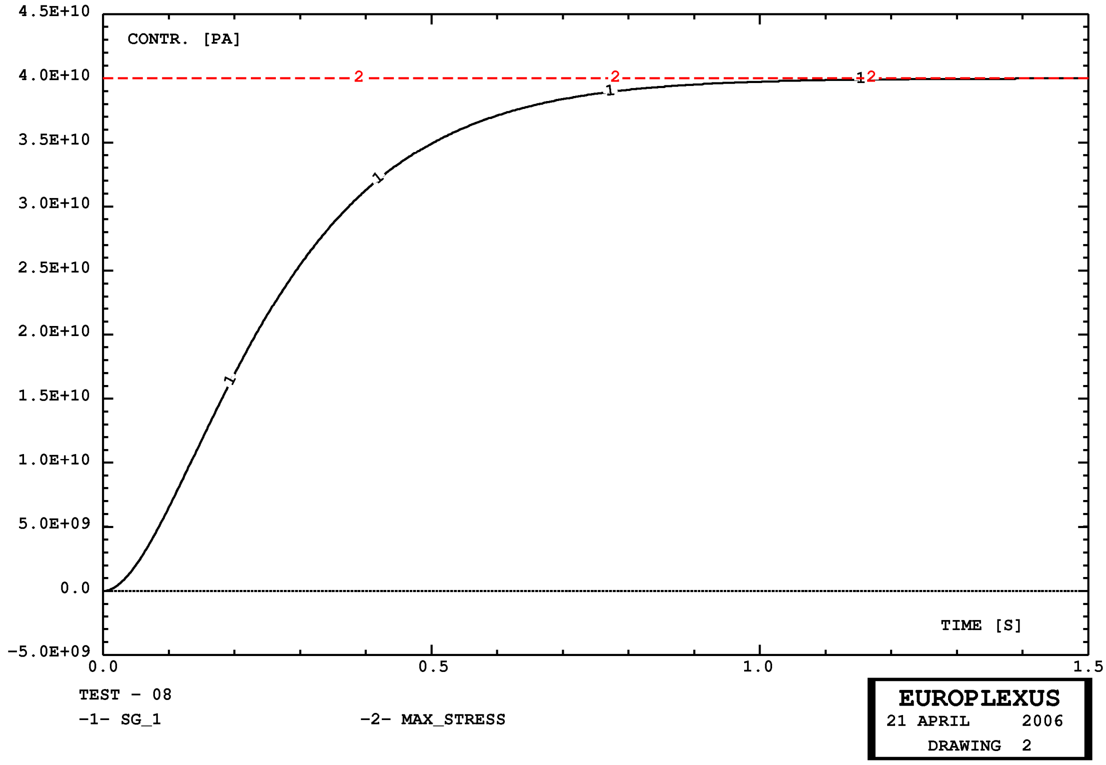
$$e = \ln \frac{L_0 + \Delta L}{L_0} = \ln(1 + \varepsilon) \qquad \text{and} \qquad \varepsilon = \exp(e) - 1$$

These functions have the following shape:

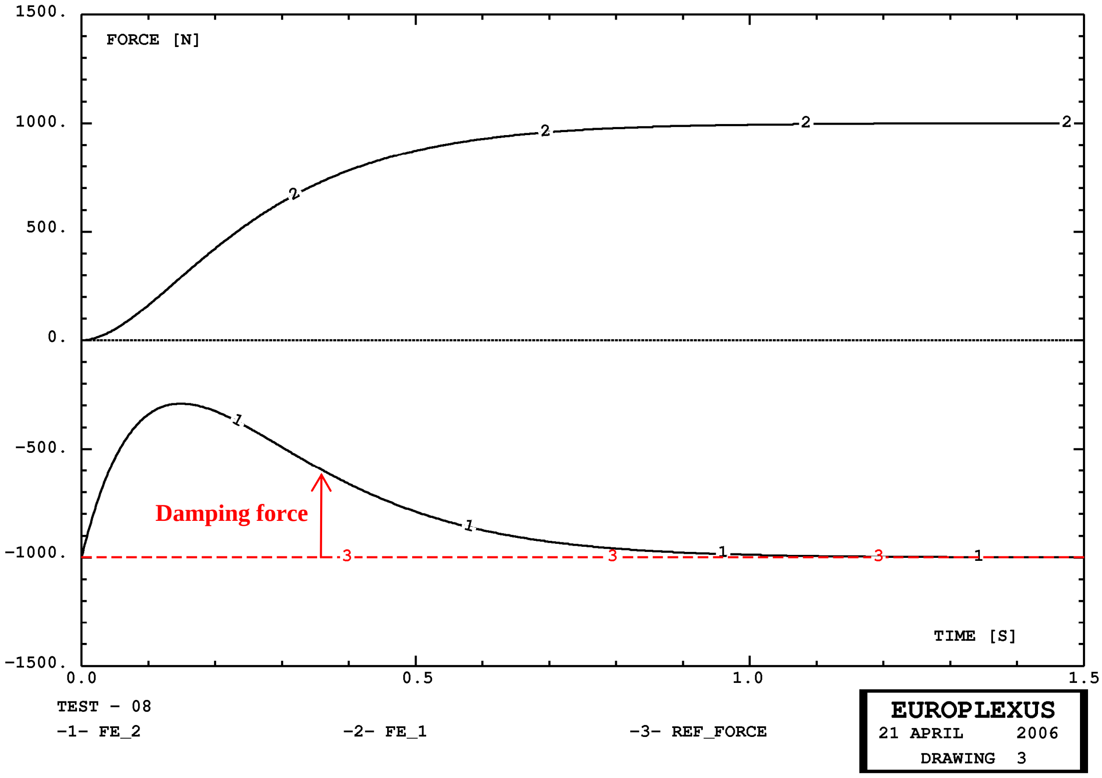
L / L_0	$\Delta L / L_0$	ε	e	Graph
0.0	-1.0	-1.0	$-\infty$	
0.5	-0.5	-0.5	-0.693	
1.0	0.0	0.0	0.000	
1.5	0.5	0.5	0.405	
2.0	1.0	1.0	0.693	



The resulting stress is 4×10^{10} as foreseen by the linear static analysis:

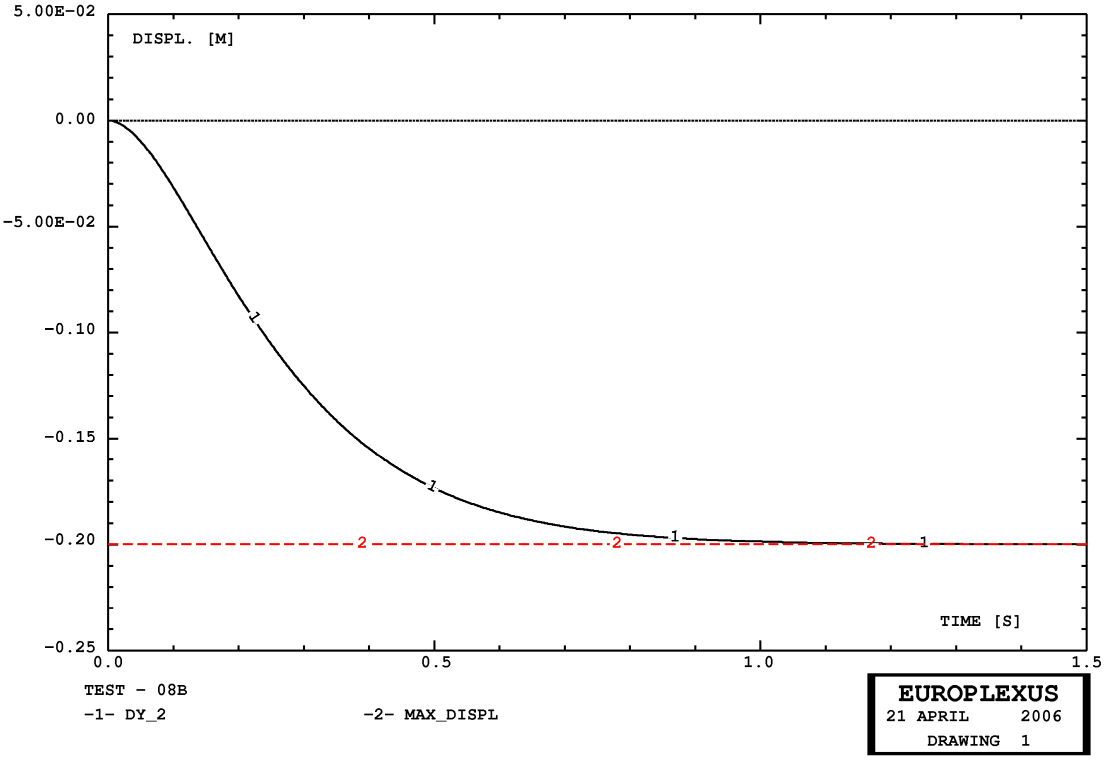


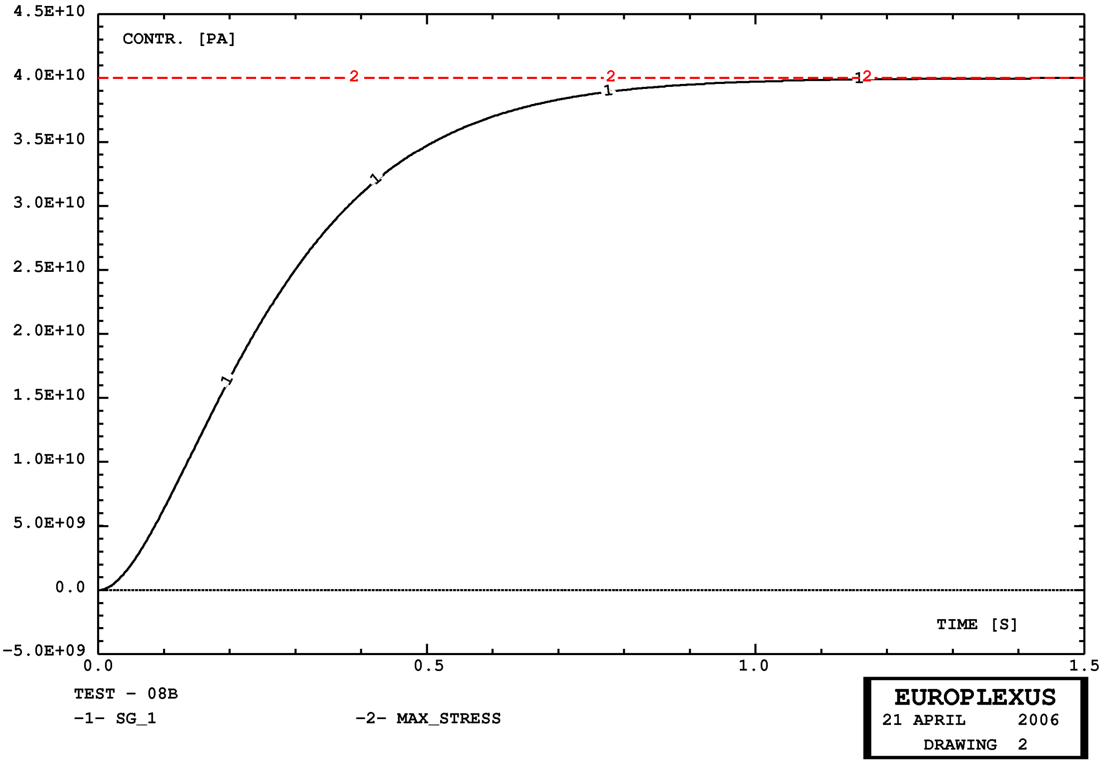
The total external forces at the two extremities have the form shown below. Note that the force applied to the moving end is the sum of the applied force (constant) and of the damping force caused by the quasi-static option, which is proportional to the velocity



TEST08B

Same as TEST08 but uses small-strain option OPTI EDSS. In that case the expected displacement is –0.2 and the expected stress is 4.E10. In this calculation we use the expected damping frequency for the linear case: $f_{sys} = 1.125$ Hz.





TEST12

Same as TEST01 but uses a time increment 100 times larger, i.e. 10 ms instead of 0.1 ms. This value violates the Courant condition, as it is readily verified. In fact in this case the element length is $L = 1$ m while the sound speed in the cable material is:

$$c = \sqrt{E/\rho} = \sqrt{2\times10^{11}/8000} = 5000 \text{ m/s}$$

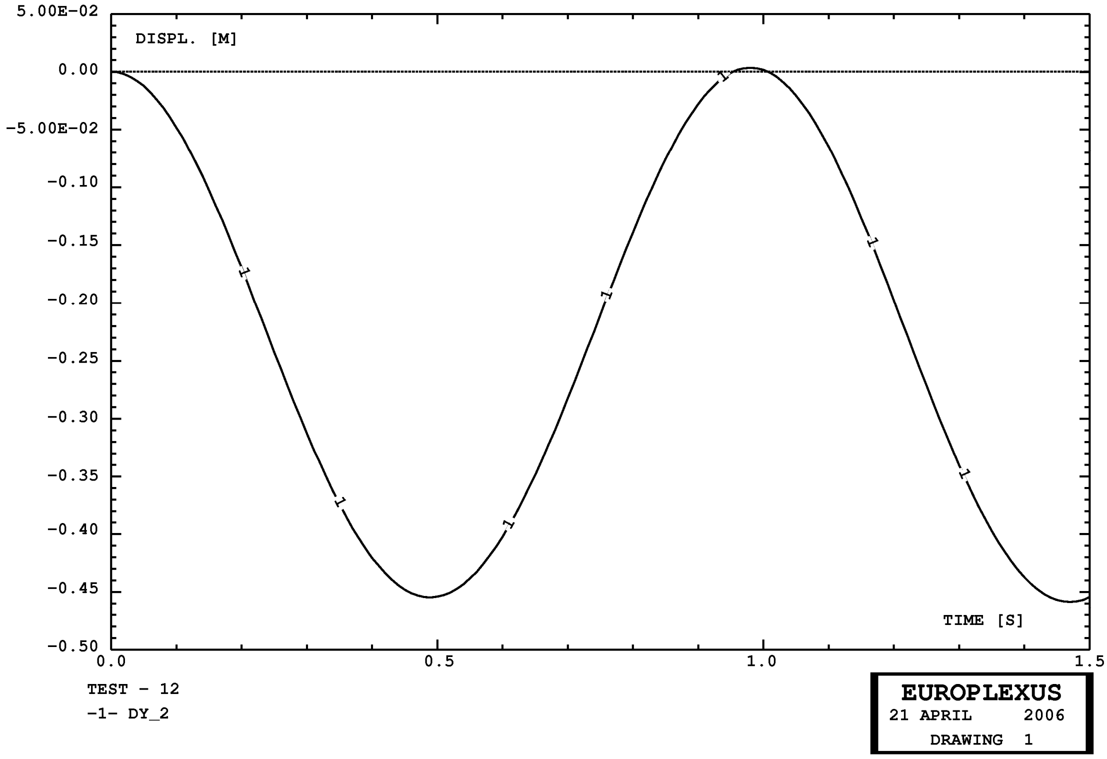
Therefore the estimated critical time increment would be:

$$\Delta t^{\mathrm{crit}} \simeq L/c = 1/5000 = 0.2 \text{ ms}$$

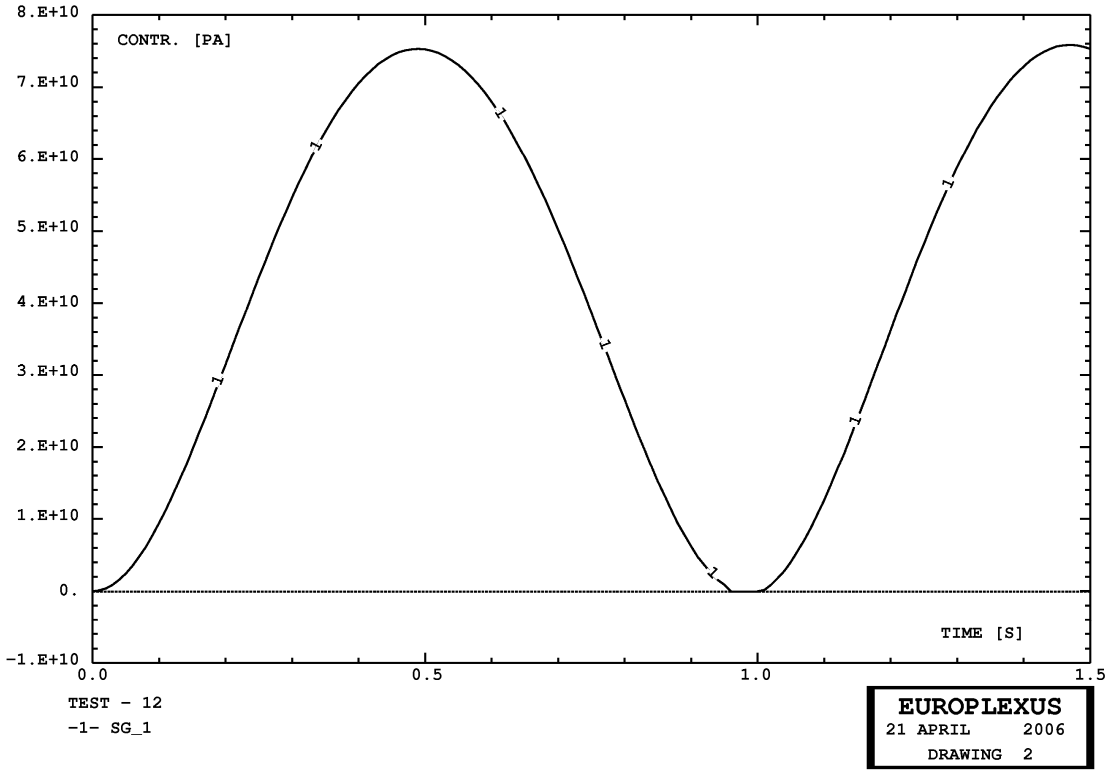
Nevertheless, if one runs the program with the excessive value of time increment, the obtained results are quite similar to those obtained with the stability-compliant value (see graphs below).

The reason is that this is a very special case. Having used just one element to “discretize” the problem geometry, there are no wave propagation phenomena in the numerical model: one node is fixed and the other one receives the applied load. Therefore, the stability condition does not apply in this case.

Try out a finer discretization, involving more than one element, to see the disastrous effect of using a time increment beyond the stability limit on the numerical solution.



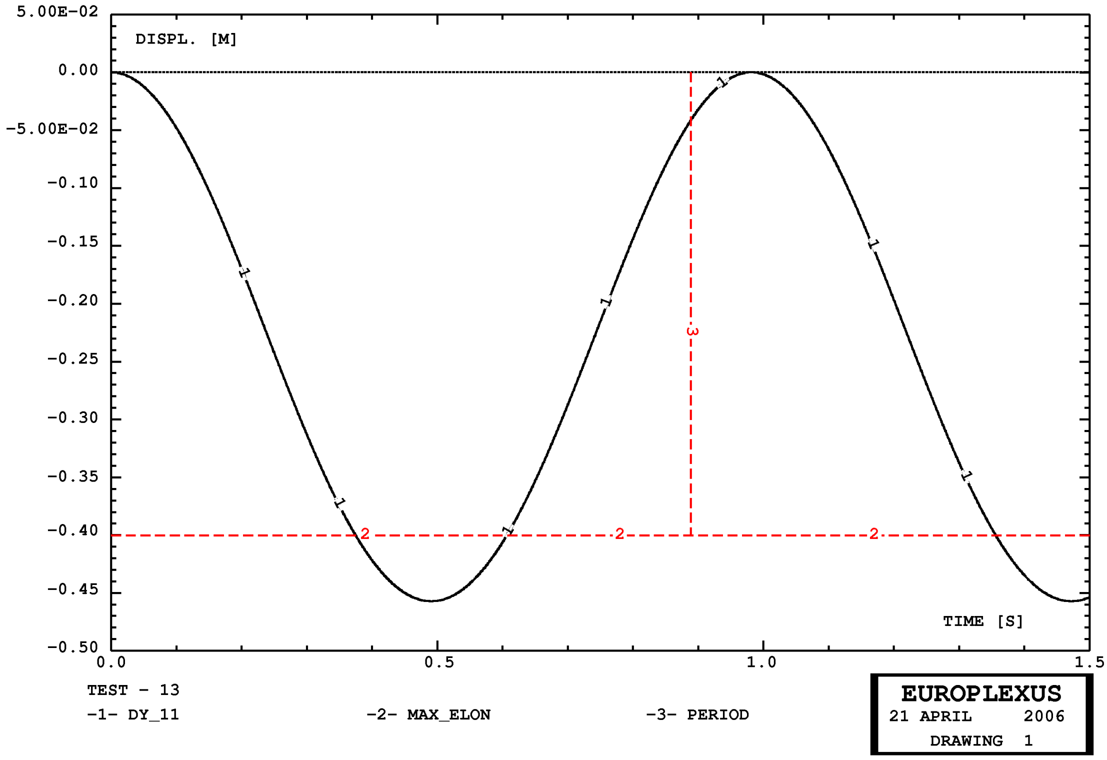
5.00E-02
0.00
-5.00E-02
-0.10
-0.15
-0.20
-0.25
-0.30
-0.35
-0.40
-0.45
-0.50
DISPL. [M]
TIME [S]
0.0
0.5
1.0
1.5
TEST - 12
-1- DY_2
EUROPLEXUS
21 APRIL 2006
DRAWING 1

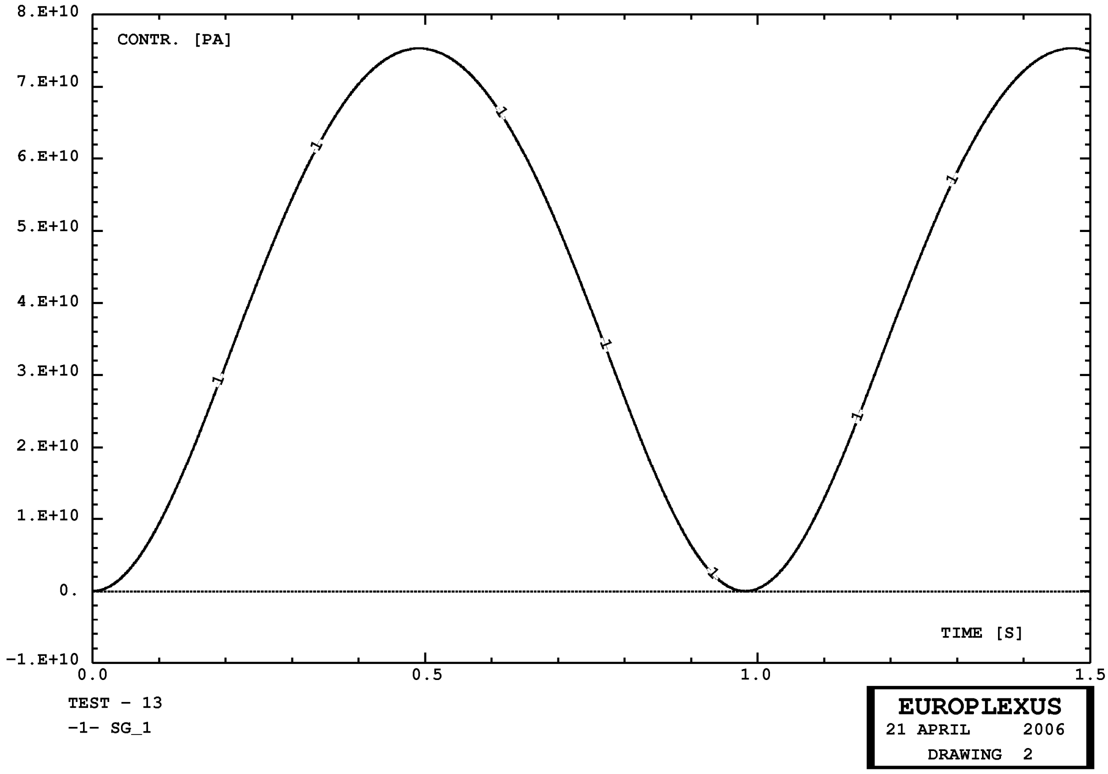


8.E+10
7.E+10
6.E+10
5.E+10
4.E+10
3.E+10
2.E+10
1.E+10
0.
-1.E+10
CONTR. [PA]
TIME [S]
0.0
0.5
1.0
1.5
TEST - 12
-1- SG_1
EUROPLEXUS
21 APRIL 2006
DRAWING 2

TEST13

Same as TEST01 but uses a 10-element mesh (Δt 10 times smaller).





TEST14
Same as TEST13 but uses same Δt as TEST01. This calculation is unstable, as expected.

TEST21
To conclude this exercise, let us consider a similar problem whereby we replace the gravity $\underline{g}$ by an initial velocity $\underline{v}_0$ directed downwards.

The initial kinetic energy of the system is:

$$E_{K0} = \frac{1}{2} M v_0^2$$

The mass will stop when all this energy has been transformed into elastic energy in the cable or bar. Assuming linear elastic behaviour and Poisson's coefficient $\nu = 0$ the resistance force of the bar is:

$$R = \sigma S = E \varepsilon S = \frac{ES}{L_0} \lambda$$

where λ is the elongation:

$$\lambda \doteq L - L_0$$

The elastic energy is:

$$E_E = \int_0^{\Delta L} \frac{ES}{L_0} \lambda d\lambda = \frac{ES}{L_0} \frac{(\Delta L)^2}{2}$$

By posing $E_E = E_{K0}$ one obtains from the above expressions:

$$\Delta L = \sqrt{\frac{L_0 M}{ES}} v_0$$

In order to obtain an elongation of, say, 20 % of the initial length:

$$\Delta L = 0.2 L_0 = 0.2 \text{ m}$$

the initial velocity should be:

$$v_0 = \sqrt{\frac{ES}{L_0 M}} \Delta L = \sqrt{\frac{2.0\times10^{11} \cdot 2.5\times10^{-8}}{1 \cdot 100}} = 1.414 \text{ m/s}$$

Let us now compute the time τ to reach the maximum elongation. The equation of motion of the system may be written as:

$$M \ddot{\lambda} = -\frac{ES}{L_0} \lambda \quad \rightarrow \quad \ddot{\lambda} = -\frac{ES}{ML_0} \lambda$$

The solution of this equation, for the particular initial conditions considered in this problem, is the following harmonic motion:

$$\lambda(t) = \sin(\omega t)$$

with the pulsation given by:

$$\omega = \sqrt{\frac{ES}{ML_0}} = 7.071 \text{ s}^{-1}$$

i.e. the same value as in the case considered previously, with gravity and zero initial velocity. The desired time τ is clearly ¼ of an oscillation period T, and therefore is given by:

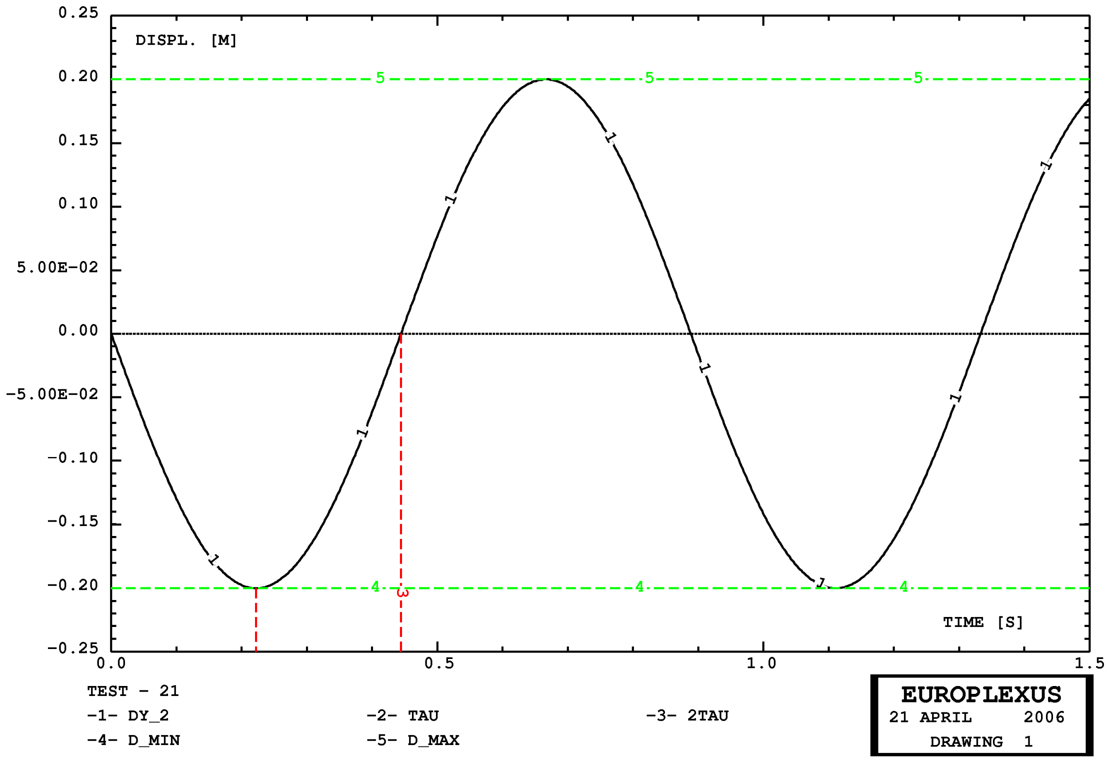
$$\tau = \frac{T}{4} = \frac{2\pi}{4\omega} = 0.222 \text{ s}$$

The mass returns at the initial position after a time $2\tau = 0.444$ s .
Note that these times do *not* depend upon the initial velocity of the mass.

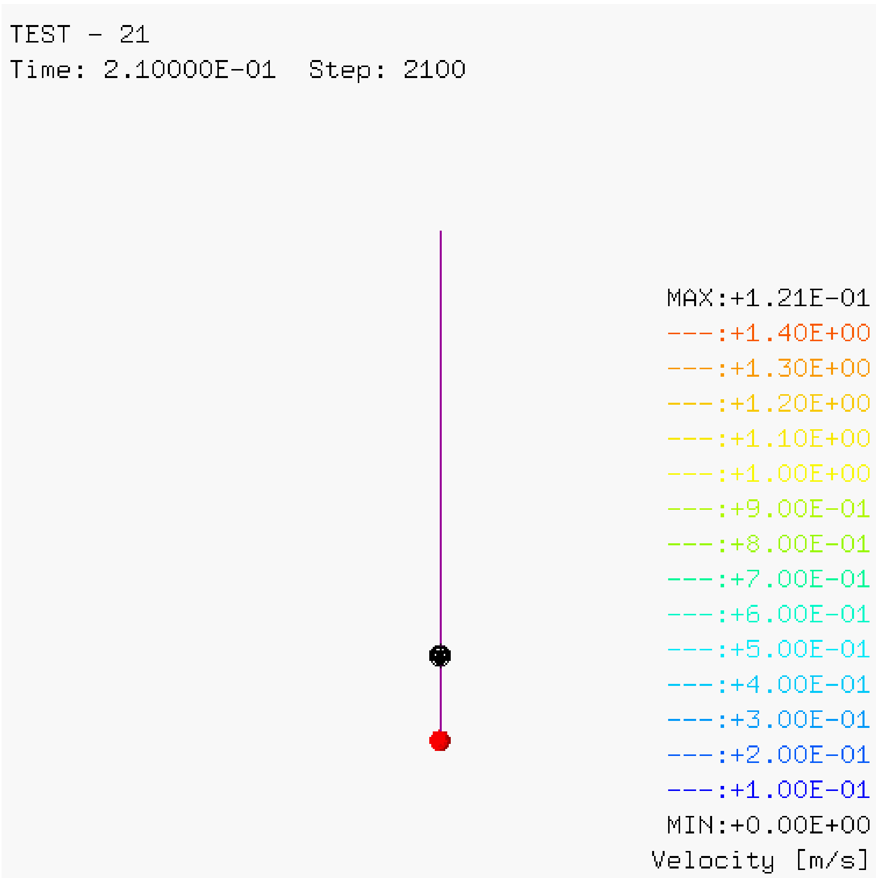
To check these analytical results, a calculation (TEST21) is performed, similar to case TEST11 i.e. by the small strain option EDSS, but without gravity and by an initial velocity of –1.414 m/s in the vertical direction. The input file is as follows:

```
TEST - 21
*----------------------------------------------------------------------
ECHO
*CONV win
*----------------------------------------------------------Problem type
CPLA NONL LAGR
*----------------------------------------------------------Dimensioning
DIME
  PT2L 2 FUN2 1 PMAT 1 ZONE 2
  TABL 1 2
  FORC 1
TERM
*--------------------------------------------------------------Geometry
GEOM LIBR POIN 2 FUN2 1 PMAT 1 TERM
 0 0  0 -1
 1 2
 2
*-----------------------------------------------Geometrical complements
COMP EPAI 2.5E-8 LECT 1 TERM
*---------------------------------------------------------Material data
MATE VM23 RO 8000. YOUN 2.0E11 NU 0.0 ELAS 2.0E11
          TRAC 1 2.0E11 1.E0
          LECT 1 TERM
     MASS 100.0 LECT 2 TERM
*---------------------------------------------------Boundary conditions
LINK COUP
     BLOQ 2 LECT 1 TERM
*----------------------------------------------------Initial conditions
INIT VITE 2 -1.4142 LECT 2 TERM
*---------------------------------------------------------------Outputs
ECRI DEPL VITE CONT ECRO TFREQ 0.5
     FICH ALIC TEMP FREQ 20
               POIN LECT 1 2 TERM
               ELEM LECT 1 TERM
*---------------------------------------------------------------Options
OPTI PAS UTIL NOTEST
     EDSS
*-------------------------------------------------Transient calculation
CALCUL TINI 0. TEND 1.5 PASF 0.1E-3
*==========================================================POST-TREATMENT
SUIT
Post-treatment
ECHO
RESU ALIC TEMP GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
*---------------------------------------------------Curve definitions
COUR 1 'dy_2'     DEPL COMP 2 NOEU LECT  2 TERM
COUR 2 'sg_1'     CONT COMP 1 ELEM LECT  1 TERM
COUR 3 'fe_2'     FORC COMP 2 NOEU LECT  2 TERM
COUR 4 'fe_1'     FORC COMP 2 NOEU LECT  1 TERM
DCOU 5 'tau' 2
       0.222 -0.20
       0.222 -0.25
DCOU 6 '2tau' 2
       0.444  0.00
       0.444 -0.25
DCOU 7 'd_min' 2
       0.   -0.20
       1.5 -0.20
DCOU 8 'd_max' 2
       0.    0.20
       1.5  0.20
*-----------------------------------------------------------------Plots
trac 1 5 6 7 8 axes 1.0 'DISPL. [M]' yzer
           colo noir roug roug vert vert
           dash 0    2    2    2    2
           noyl 0    1    1    1    1
trac 2 axes 1.0 'CONTR. [PA]' yzer
trac 3 4 axes 1.0 'FORCE [N]' yzer
*-------------------------------------------------Results qualification
QUAL DEPL COMP 2 LECT   2 TERM REFE  1.85058E-01 TOLE 1.E-2
     CONT COMP 1 LECT   1 TERM REFE -3.70116E+10 TOLE 1.E-2
*======================================================================
FIN
```

The resulting displacement is shown next and is in excellent agreement with the expected value:

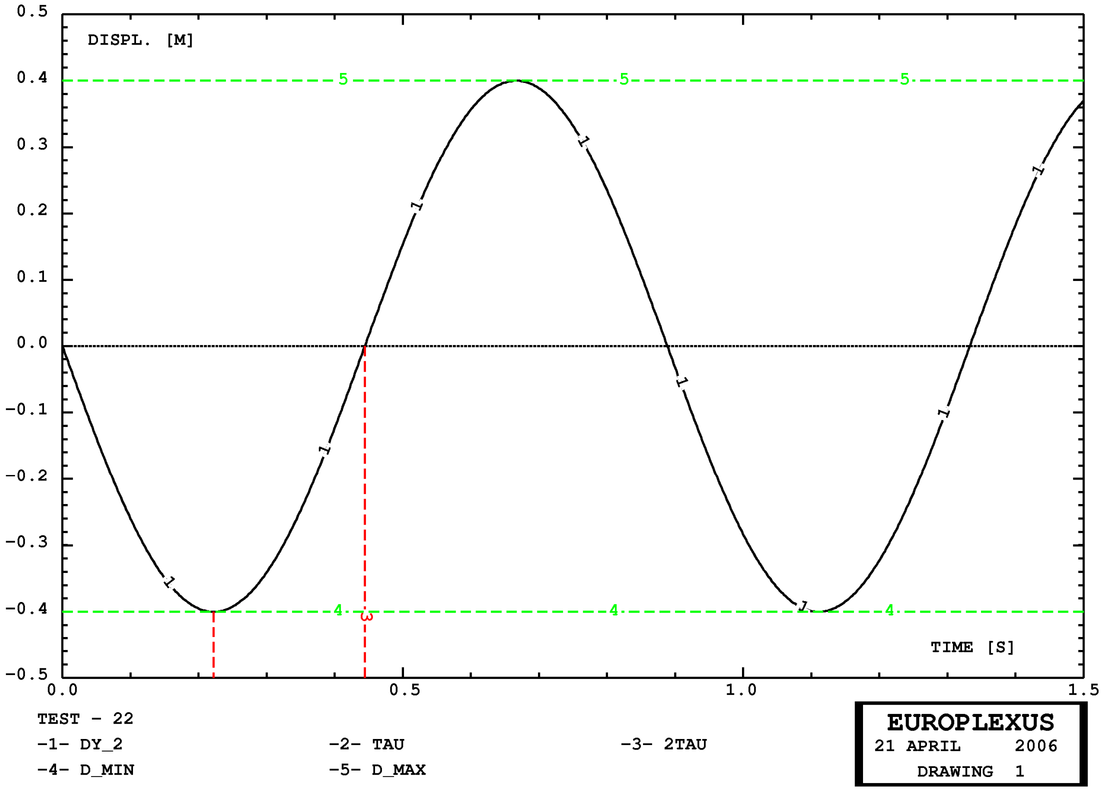


Here is an animation of results:



TEST22

To verify the independence of oscillation period from the initial velocity, we repeat the same calculation but with a twice larger initial velocity $v_0 = -2.828 \text{ m/s}$. The result is shown next and has the expected shape (the elongation is twofold):

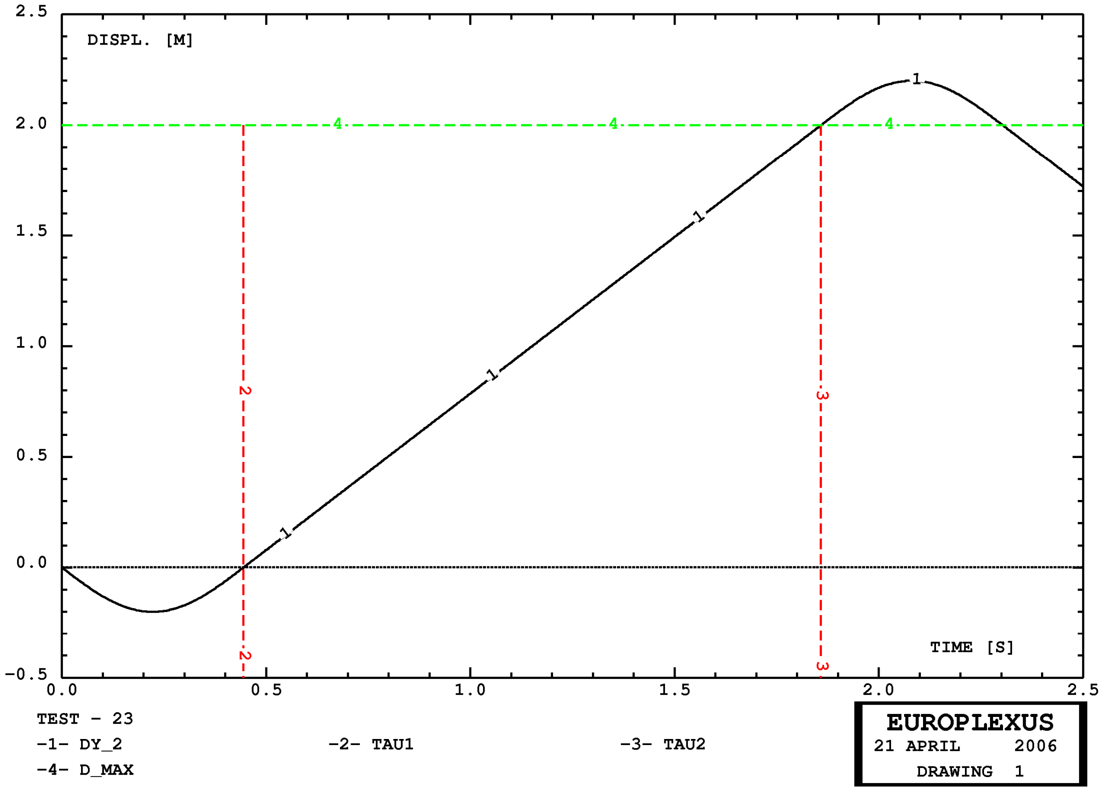


TEST23

In the previous two examples a material of type VM23 (Von Mises isotropic) was assigned to the cable, which therefore acted as a bar (same resistance in compression as in traction).

If one assumes a real cable (FUNE material), however, the behaviour will be different. Since there is no resistance to compression, the mass will have to bounce twice the initial cable length before exerting a traction (in the direction opposite to the initial one) on the cable.

This behaviour is simulated in the calculation TEST23. The picture below shows the displacement in this case:



The load is reversed when the cable has reached the opposite orientation. The "load-free" period is:

$$2L / v_0 = (2 \cdot 1) / 1.4142 = 1.4142 \text{ s}$$

This result is fully confirmed by the numerical calculation, as may be verified graphically from the figure above.