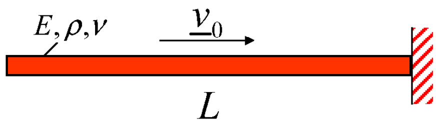
Exercise 2 – Wave propagation

- Obtain 1D analytical solution
- Discuss numerical solution
- Why was cross-section not specified?
- Study effect of time increment
- Study effect of Poisson's ratio …



$L = 1$ m
$\underline{v}_0 = 100$ m/s
$E = 2\times10^{11}$ Pa
$\rho = 8000$ kg/m^3
$\nu = 0.3$

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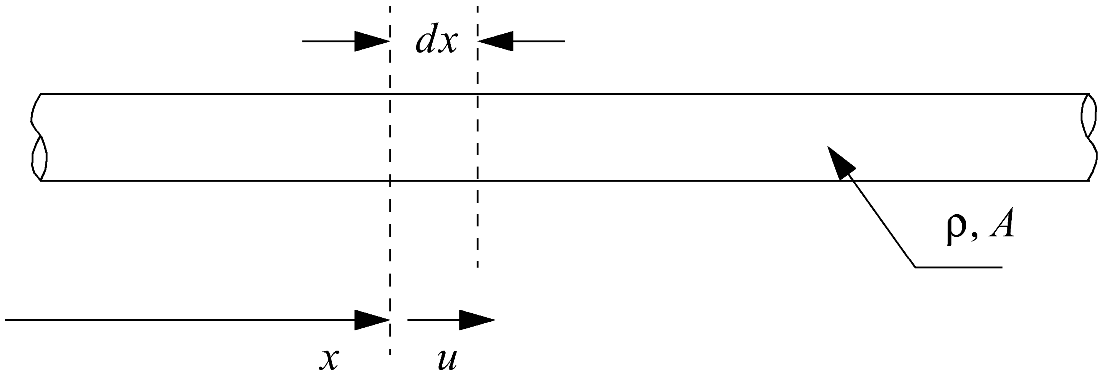
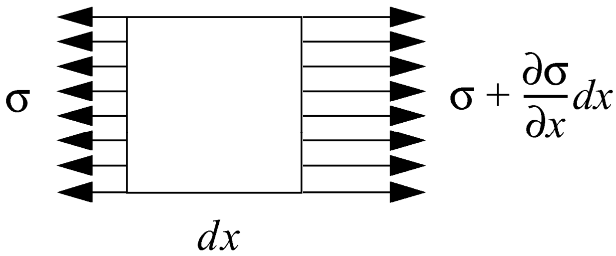
<u>Linear dynamic analysis</u>

Consider the longitudinal waves in a long bar subjected to axial loading.

Assume that:

- The cross section A is constant;
- The material, of density ρ, is homogeneous and isotropic;
- Plane, parallel cross sections remain plane and parallel;
- The stress σ is uniform in each cross section;
- We neglect the effect of lateral inertia.





The (dynamic) equilibrium is expressed by the following equation (equation of motion):

$$-\sigma A + \left(\sigma + \frac{\partial \sigma}{\partial x} dx \right) A = \rho A dx \frac{\partial^2 u}{\partial t^2}$$

from which we obtain:

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}$$

For an elastic material we have $\sigma = E\varepsilon$, and since the longitudinal deformation is $\varepsilon = \partial u / \partial x$, we may re-write the last equation as:

$$E \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2}$$

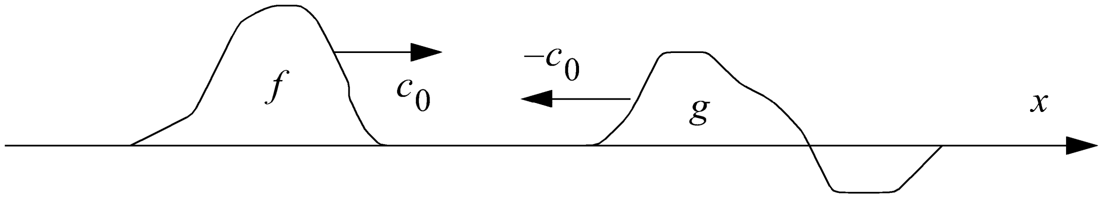
or, finally:

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{with} \qquad c_0 = \sqrt{E / \rho}$$

This is known as the (1-D) <u>wave equation</u> and the constant c_0 is the sound speed in the elastic material.

The general solution (D'Alembert's solution) to this equation reads:

$$u(x,t) = f(x - c_0 t) + g(x + c_0 t)$$



- The two waves f and g propagate without distortion
- The spatial form of f and g is determined by initial conditions and by boundary conditions

While the waves propagate at (constant) velocity c_0, the material particles move at velocity:

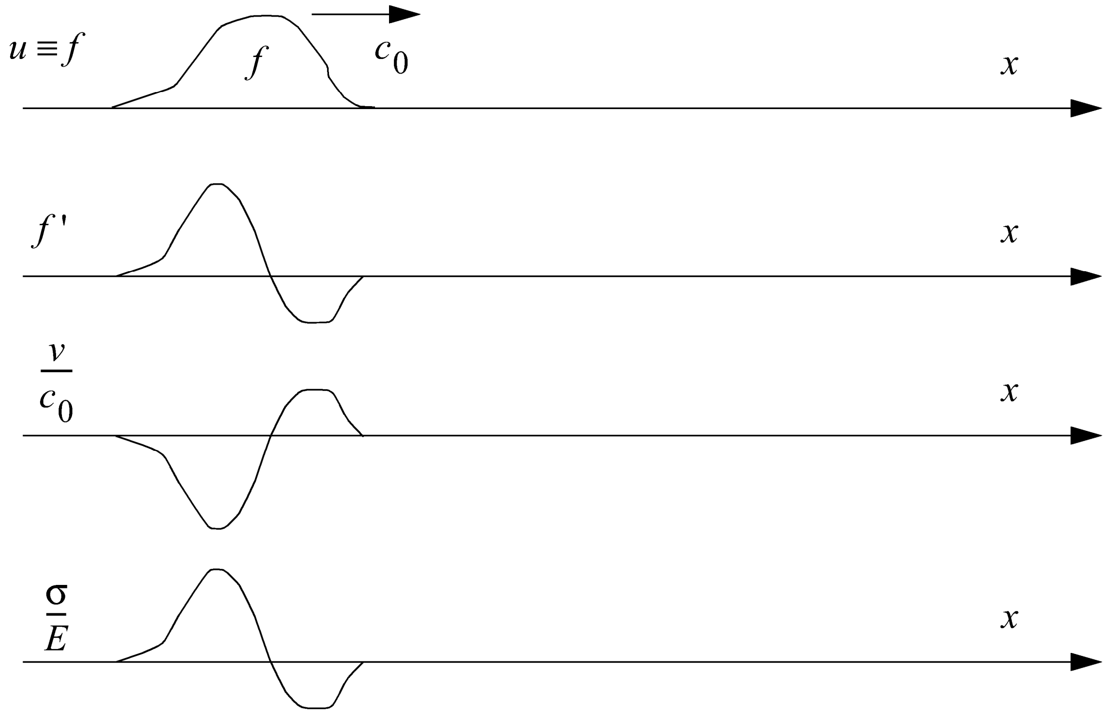
$$v(x,t) = \frac{\partial u}{\partial t} = -c_0 \frac{\partial f}{\partial t} = -c_0 f'(x - c_0 t) \qquad \text{with} \qquad f'(x - c_0 t) \doteq \frac{\partial f(x - c_0 t)}{\partial (x - c_0 t)}$$

But since:

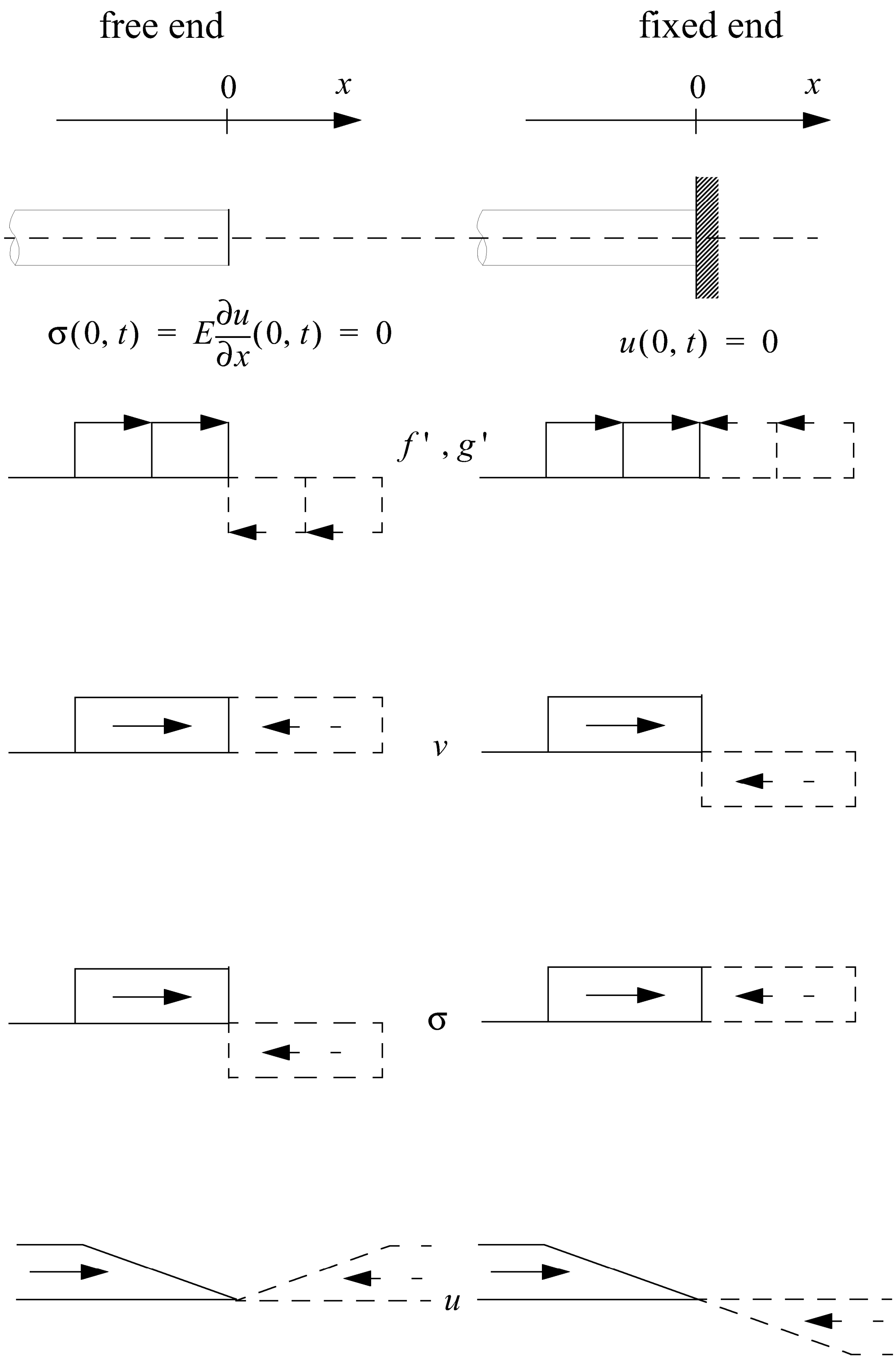
$$\sigma(x,t) = E\varepsilon = E \frac{\partial u}{\partial x} = Ef'(x - c_0 t)$$

we have:

$$v(x,t) = -\frac{c_0}{E} \sigma(x,t)$$



Simple 1-D wave propagation problems may be studied quite effectively by the <u>method of images</u>:



Analytical solution

For the bar impact problem proposed above, the analytical solution according to the linear 1-D wave propagation theory is as follows.

The sound speed in the material is:

$$c_0 = \sqrt{E/\rho} = \sqrt{2\times10^{11}/8000} = 5000 \text{ m/s}$$

From the relationship $v = -(c_0/E)\sigma$ we obtain for the stress:

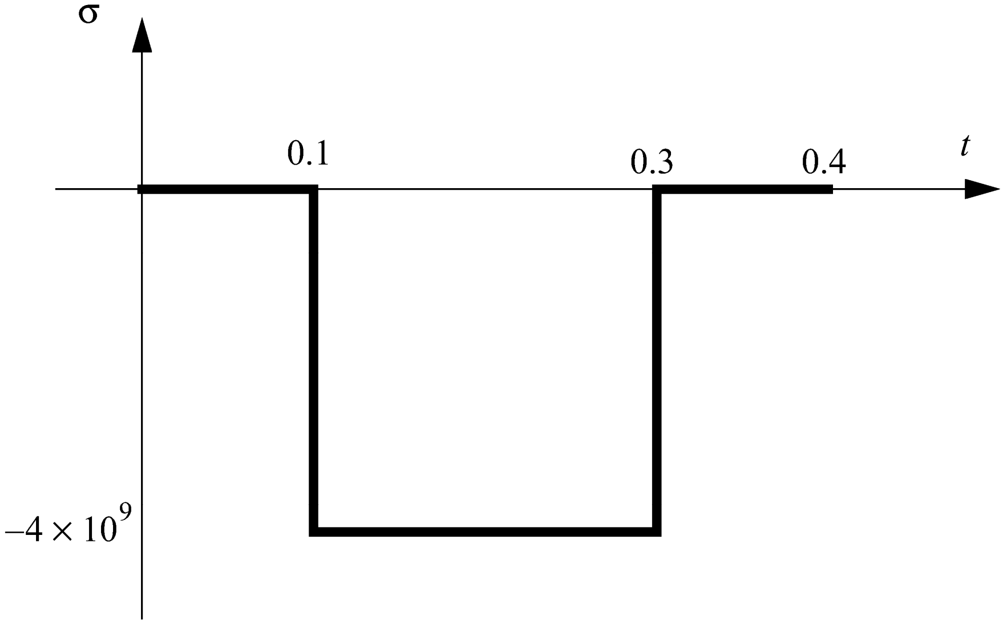
$$\sigma = -\frac{v}{c_0}E = -\frac{100}{5000}2\times10^{11} = -4\times10^{9}$$

and the longitudinal strain is:

$$\varepsilon = \sigma/E = 4\times10^{9}/2\times10^{11} = 0.02$$

- The impact against the rigid obstacle produces a step-like stress wave that enters the bar from the right end and travels towards the left at speed c_0.
- When the compression wave, at time $t_1 = L/c_0 = 1/5000 = 0.2$ ms, reaches the left end, which is free, it is reflected as a traction wave that moves to the right at the same speed and cancels out the compression.
- At time $t_2 = 2L/c_0 = 0.4$ ms the tension wave reaches the right end of the bar: the bar is stress-free and it starts rebounding towards the left with a velocity $-v$, i.e. opposite to the impact velocity.

The time history of longitudinal stress at the mid-point of the bar is therefore as follows:



Numerical Solutions

BARI01

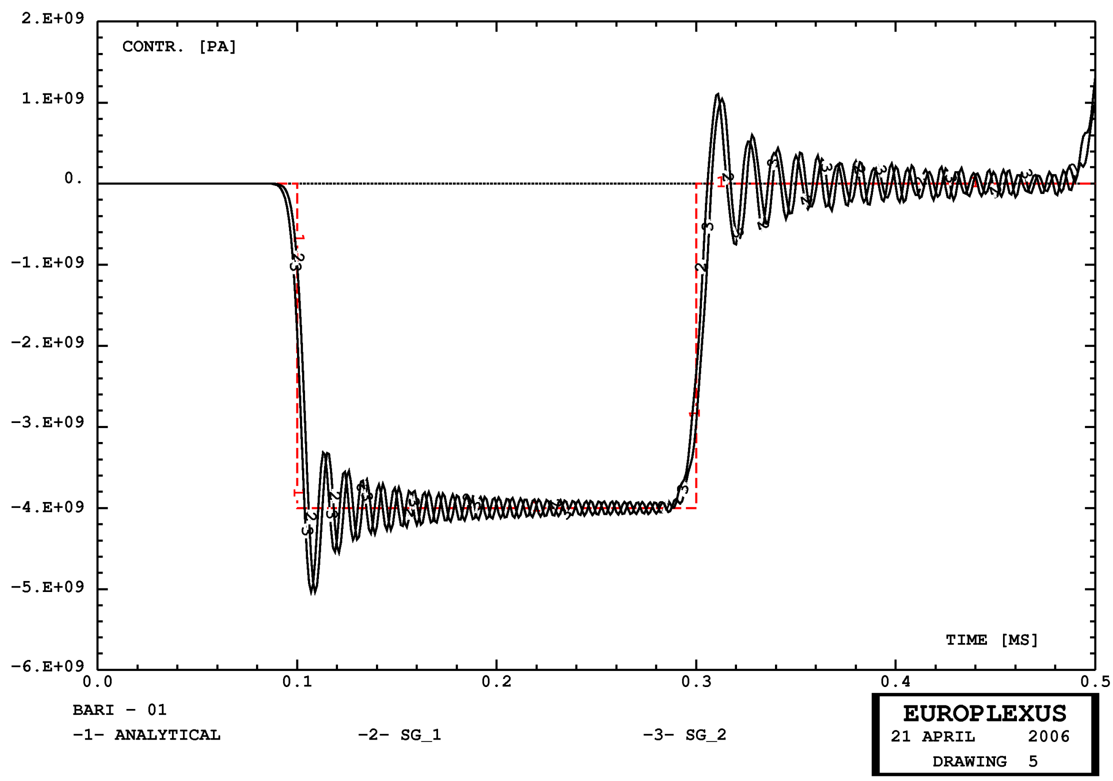
We discretize the bar with 100 elements of the FUN2 type, with an elastic material of type VM23 (resisting to both traction and compression) and use the `OPTI EDSS` option to impose small-strain formulation in the elements (see Exercise 1).

The EUROPLEXUS input file is:

```
BARI - 01
*------------------------------------------------------------------------
ECHO
*CONV win
CAST mesh
*------------------------------------------------------------Problem type
CPLA NONL LAGR LAGC
*------------------------------------------------------------Dimensioning
DIME
  PT2L 102 FUN2 100 PMAT 1 ZONE 2
  BLOQ 2
  TABL 1 2
  FORC 1
  IMPA 1 PSIM 1
TERM
*----------------------------------------------------------------Geometry
GEOM FUN2 bar PMAT obst TERM
*------------------------------------------------Geometrical complements
COMP EPAI 2.5E-8 LECT bar TERM
*-----------------------------------------------------------Material data
MATE VM23 RO 8000. YOUN 2.0E11 NU 0.0 ELAS 2.0E11
          TRAC 1 2.0E11 1.E0
          LECT bar TERM
     MASS 1.0 LECT obst TERM
*-----------------------------------------------------Boundary conditions
LIAI freq 1
     BLOQ 12 LECT obst TERM
     IMPA DDL 1 COTE -1
          PROJ LECT obst TERM
          CIBL LECT p2   TERM
*------------------------------------------------------Initial conditions
INIT VITE 1 100.0 LECT bar TERM
*-----------------------------------------------------------------Outputs
ECRI DEPL VITE CONT ECRO TFREQ 0.1E-3
     FICH ALIC TEMP FREQ 1
               POIN LECT p3 p2 TERM
               ELEM LECT e1 e2 TERM
*-----------------------------------------------------------------Options
OPTI PAS UTIL NOTEST
     LOG 1
     EDSS
*-----------------------------------------------------Transient calculation
CALCUL TINI 0. TEND 0.5E-3 PASF 0.1E-5
*===============================================================POST-TREATMENT
SUIT
Post-treatment
ECHO
RESU ALIC TEMP GARD PSCR
SORT GRAP
AXTE 1000.0 'Time [ms]'
*---------------------------------------------------------Curve definitions
COUR 1 'dx_2'     DEPL COMP 1 NOEU LECT p2 TERM
COUR 2 'dx_3'     DEPL COMP 1 NOEU LECT p3 TERM
COUR 3 'sg_1'     CONT COMP 1 ELEM LECT e1 TERM
COUR 4 'sg_2'     CONT COMP 1 ELEM LECT e2 TERM
DCOU 5 'Analytical' 6
       0.0    0.0
       0.1E-3 0.0
       0.1E-3 -4.E9
       0.3E-3 -4.E9
       0.3E-3 0.0
       0.5E-3 0.0
*-------------------------------------------------------------------Plots
trac 1 axes 1.0 'DISPL. [M]'
trac 2 axes 1.0 'DISPL. [M]'
trac 3 axes 1.0 'CONTR. [PA]'
trac 4 axes 1.0 'CONTR. [PA]'
trac 5 3 4 axes 1.0 'CONTR. [PA]' yzer
           colo roug noir noir
           dash 2    0    0
*-----------------------------------------------------Results qualification
QUAL DEPL COMP 1 LECT  p2 TERM REFE -9.76759E-3  TOLE 1.E-2
     DEPL COMP 1 LECT  p3 TERM REFE -9.86359E-3  TOLE 1.E-2
     CONT COMP 1 LECT  e1 TERM REFE  1.00128E+9  TOLE 1.E-2
     CONT COMP 1 LECT  e2 TERM REFE  1.31288E+9  TOLE 1.E-2
*========================================================================
FIN
```

Note that we have also put the Poisson's coefficient ν to 0 in order to produce a numerical solution as close as possible to the linear 1-D wave theory (neglect the lateral inertia effect).

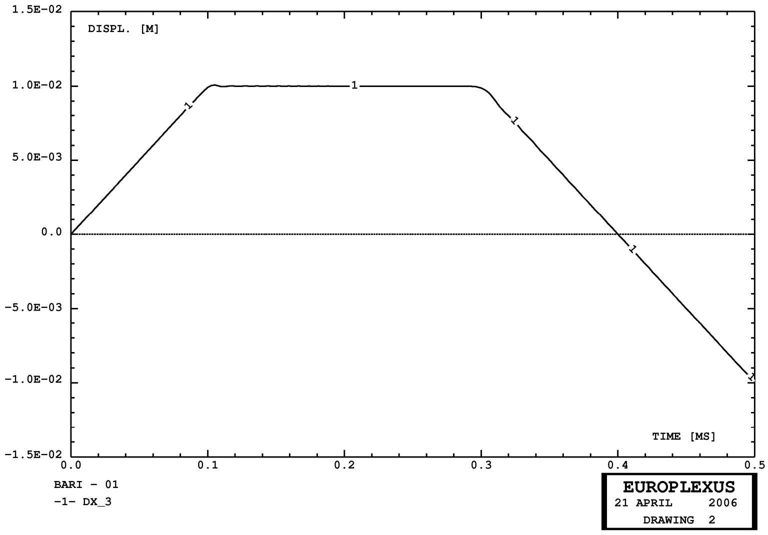
The resulting stress at the bar mid-point is:



The analytical solution is superposed (red line). Two lines are drawn for the numerical solution: they correspond to the elements immediately to the left and immediately to the right of the bar mid-point.
Note the relatively strong oscillations: recall that the central difference time integration scheme introduces no numerical damping.
The displacement of the bar central point is shown next. Note the rebound, with approximately the same velocity as the incident one.
As concerns the bar cross section, the 1-D response is independent from this parameter. We have assumed an arbitrary value here (2.5×10^{-8} , like in Example 1), because the numerical solution does not depend on this value (verify).

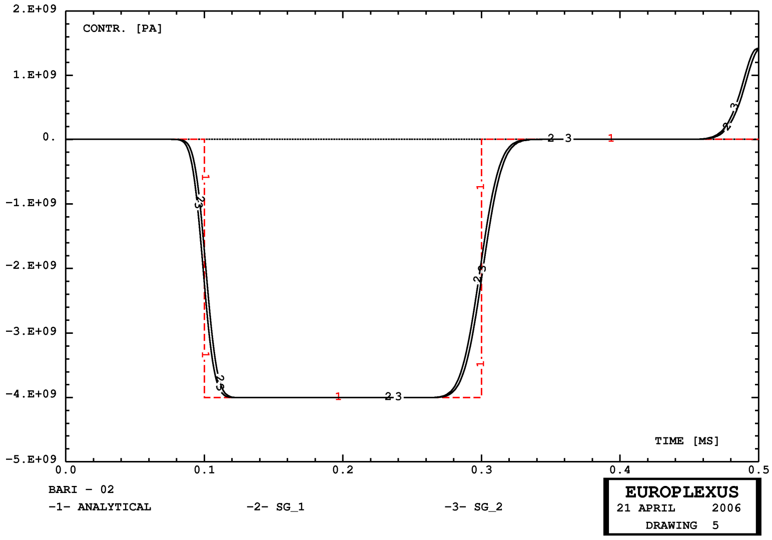
BARI01_B

Same as BARI01 but with a cross-section of 1.0: results in terms of stresses, displacements etc. are identical.



BARI02

We introduce some damping (10% of the critical value: `OPTI AMOR LINE 0.1`) to reduce the oscillations in the numerical solution. However, this has also an effect on wavefronts, which become less steep and thus deviate more from the analytical solution. The resulting stress at the bar mid-point is:

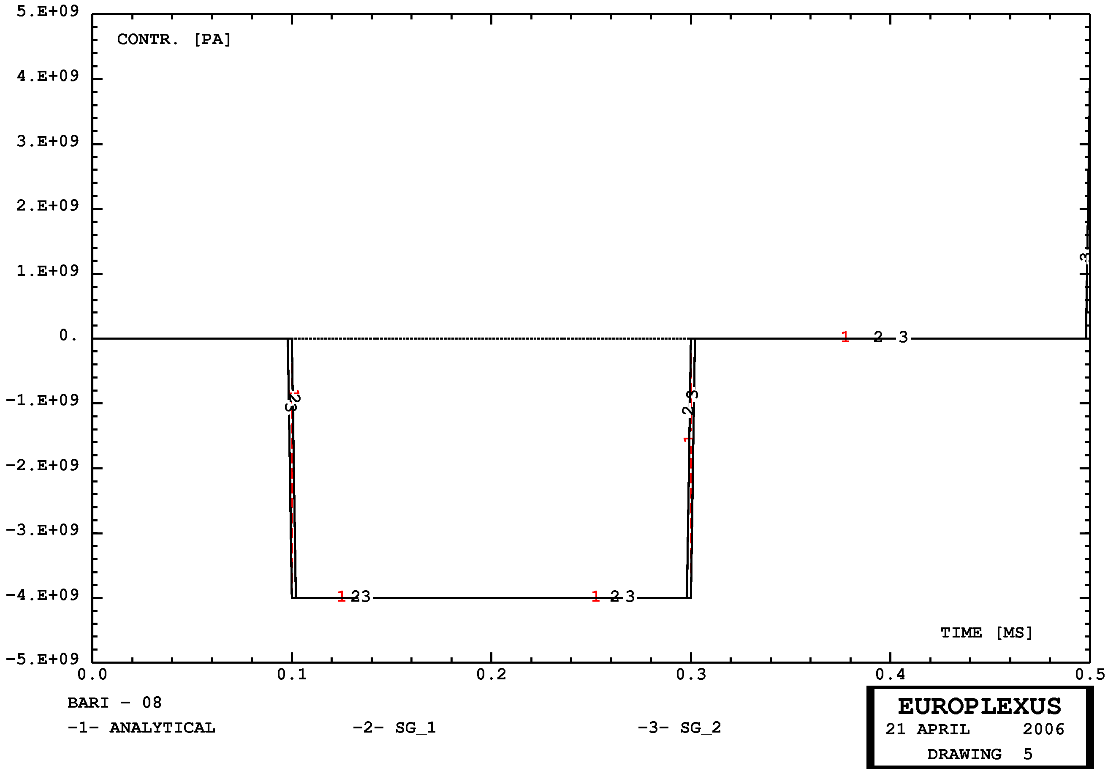


BARI08

We study the effect of the time increment. The critical time step for the chosen problem is:

$$\Delta t^{\text{crit}} \simeq \frac{L}{\sqrt{E/\rho}} = \frac{1/100}{5000} = 2\times10^{-6}\ \text{s}$$

In the previous runs we have assumed a fixed time increment $\Delta t = 1\times10^{-6} = 0.5\cdot\Delta t^{\text{crit}}$. Let us see what happens by taking a time increment larger than the estimated stability limit. By assuming $\Delta t = 4\times10^{-6} = 2.0\cdot\Delta t^{\text{crit}}$, the numerical solution “explodes” (∞ velocities) after just 10 time steps (try out to see).
By assuming $\Delta t = 2\times10^{-6} = 1.0\cdot\Delta t^{\text{crit}}$, the numerical solution is carried out until the end and the results are very close to the analytical solution:



Note that this calculation is done <u>without</u> any added damping. The fact that an almost analytical result is obtained is remarkable, but it should be noted that in practice using a time step equal to the critical value is impossible for real calculations where the mesh size and/or the material properties change in space and in time.
It is always advisable to use a time increment as close as possible to the stability limit, but by bearing in mind that the latter is often just an estimation.

BARI08_B

Same as BARI08 but twice larger Δt: the calculation is unstable as expected.

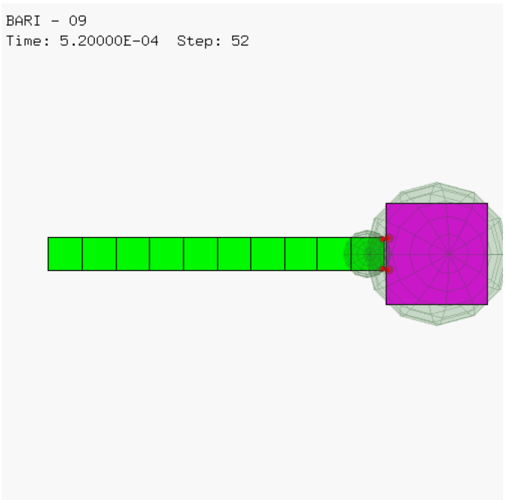
BARI09

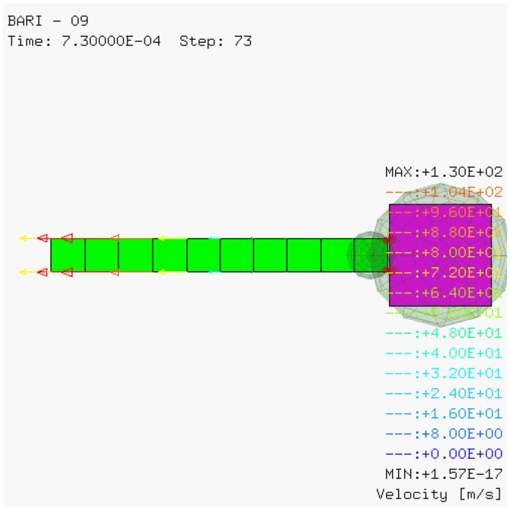
Use a 2-D geometry (elements of type Q42L). The solution is independent upon the extension along the y-direction.

Use the pinball model to describe the impact (see Part IV). The input file is:

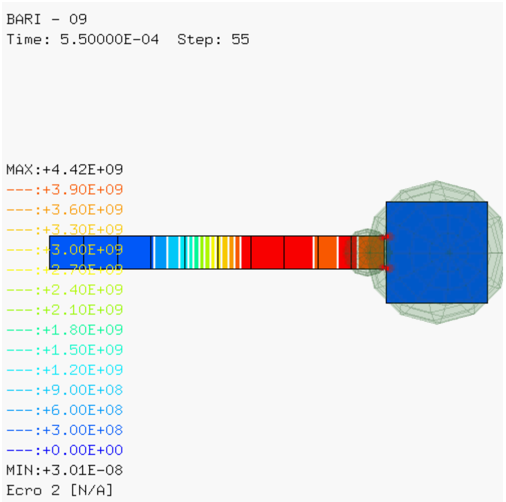
```
BARI - 09
*----------------------------------------------------------------------
ECHO
 CONV win
CAST mesh
*------------------------------------------------------------Problem type
CPLA NONL LAGR
*------------------------------------------------------------Dimensioning
DIME
  PT2L 26 Q42L 11
  TABL 1 2
  FORC 1
TERM
*----------------------------------------------------------------Geometry
GEOM Q42L bar obst TERM
*-----------------------------------------------Geometrical complements
COMP EPAI 2.5E-8 LECT bar obst TERM
*-----------------------------------------------------------Material data
MATE VM23 RO 8000. YOUN 2.0E11 NU 0.0 ELAS 2.0E11
          TRAC 1 2.0E11 1.E0
          LECT bar obst TERM
*---------------------------------------------------Boundary conditions
LINK COUP
     BLOQ 12 LECT obst TERM
     PINB BODY MLEV 3 LECT bartip TERM
          BODY MLEV 4 LECT obst   TERM
*----------------------------------------------------Initial conditions
INIT VITE 1 100.0 LECT bar TERM
*-----------------------------------------------------------------Outputs
ECRI DEPL VITE CONT ECRO TFREQ 0.1E-3
     FICH ALIC TEMP FREQ 1
               POIN LECT p3 p2 TERM
               ELEM LECT e1 e2 TERM
     FICH ALIC FREQ 1
*-----------------------------------------------------------------Options
OPTI PAS UTIL NOTEST
     LOG 1
     PINS CNOR
*-------------------------------------------------Transient calculation
CALCUL TINI 0. TEND 1.2E-3 PASF 0.1E-4
*=====================================================Direct Animation
PLAY

CAME   1 EYE   6.55000E-01  5.00000E-02  3.35978E+00
!        Q     1.00000E+00  0.00000E+00  0.00000E+00  0.00000E+00
         VIEW  0.00000E+00  0.00000E+00 -1.00000E+00
         RIGH  1.00000E+00  0.00000E+00  0.00000E+00
         UP    0.00000E+00  1.00000E+00  0.00000E+00
         FOV   2.48819E+01

scen geom navi free
          pinb pare cdes
     !vect scco scal user prog 0.e0 pas 0.16e2 0.80e2 term
     !text vsca
     iso  fili fiel ecro 2 scal user prog 0.e0 pas 0.3e9 3.6e9 term
     text isca
     colo pape

freq 1

sler cam1 1        nfra 1
trac offs fich avi      nocl nfto 121 fps 10 kfre 10 comp -1 rend
gotr loop 119 offs fich avi cont nocl                          rend
go
trac offs fich avi cont                                        rend

ENDPLAY
*=======================================================POST-TREATMENT
SUIT
Post-treatment
ECHO
RESU ALIC TEMP GARD PSCR
SORT GRAP
AXTE 1000.0 'Time [ms]'
*-----------------------------------------------------Curve definitions
COUR 1 'dx_2'     DEPL COMP 1 NOEU LECT p2 TERM
COUR 2 'dx_3'     DEPL COMP 1 NOEU LECT p3 TERM
COUR 3 'sg_1'     CONT COMP 1 ELEM LECT e1 TERM
COUR 4 'sg_2'     CONT COMP 1 ELEM LECT e2 TERM
*-------------------------------------------------------------------Plots
trac 1 axes 1.0 'DISPL. [M]'
trac 2 axes 1.0 'DISPL. [M]'
trac 3 axes 1.0 'CONTR. [PA]'
trac 4 axes 1.0 'CONTR. [PA]'
trac 3 4 axes 1.0 'CONTR. [PA]'
*-------------------------------------------------Results qualification
QUAL DEPL COMP 1 LECT  p2 TERM REFE  9.28567E-3  TOLE 1.E-2
     DEPL COMP 1 LECT  p3 TERM REFE  1.08357E-2  TOLE 1.E-2
     CONT COMP 1 LECT  e1 TERM REFE  8.57355E+8  TOLE 2.E-2
     CONT COMP 1 LECT  e2 TERM REFE  6.61260E+7  TOLE 1.E-2
*=====================================================================
FIN
```

This produces an animation of the results. Either the geometry or the velocities:





or the Von Mises stress:



To change the type of animated results, comment/uncomment the appropriate lines in the "`scen`" directive, near the end of the input file.

BARI10

Same as BARI09 but compares two bars: the first one is elastic like in the previous example, but the second one is elasto-plastic with a yield stress of 2×10^{9} Pa (corresponding to 1% axial strain) and a low plastic modulus.

Compare the two solutions and discuss the results: the Von Mises stress and the current yield stress